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A GUIDANCE AND NAVIGATION SYSTEM FOR CONTINUOUS LOW THRUST VEHICLES

BY
CHARLES JACK - CHING TSE

SEPTEMBER, 1973

ERRATA SHEET

p. 10 $\underline{n}/|\underline{v}|$ should be $\underline{n}/|\underline{u}|$ in the definition of n_1, n_2, n_3

p.20 $p = \frac{\omega_x}{1+\omega_z}$ should read $p = \frac{w_x}{1+w_z}$

$q = \frac{\omega_y}{1+\omega_z}$ should read $q = \frac{w_y}{1+w_z}$

p.40 numerators of the fractions in the first eqation should be 3 instead of 1

A GUIDANCE AND NAVIGATION SYSTEM FOR CONTINUOUS LOW-THRUST VEHICLES

by

Charles Jack-Ching Tse

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ABSTRACT

A Midcourse guidance and navigation system for continuous low thrust vehicles is developed in this research. The vehicle is required to reach an allowable region near the desired geosynchronous orbit from a near earth orbit in minimum time. The angular position of the vehicle in the orbit is assumed to be unimportant during this midcourse flight. The magnitude of the thrust acceleration is assumed to be bounded. The effects of the uncertainties due to the random initial state, the random thrusting error and the sensor error are included.

A set of orbit elements, known as the equinoctial elements, are selected as the state variables. The uncertainties are modelled statistically by random vector and stochastic processes. The motion of the vehicle and the measurements are described by nonlinear stochastic differential and difference equations respectively.

A minimum time nominal trajectory is defined and the equation of motion and the measurement equation are linearized about this nominal trajectory. An exponential cost criterion is constructed and a linear feedback guidance law is derived to control the thrusting direction of the engine. Using this guidance law, the vehicle will fly in a trajectory neighboring the nominal trajectory. The extended Kalman filter is used for state estimation.

Finally a short mission using this system is simulated. The results indicated that this system is very efficient for short missions. For longer missions some more accurate ground based measurements and nominal trajectory updates must be included.

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LIST OF SYMBOLS

a	Semi-major Axis
A	$\frac{\partial}{\partial \underline{x}} G \underline{u}$
$\underline{b}$	$\underline{x} - \tilde{\underline{x}}$
B	G
d	Diameter of Earth
e	Eccentricity
$\underline{e}$	Eccentricity Vector
$\exp \{\cdot\}$	Exponential
$E\{\cdot\}$	Expectation
$\underline{f}$	Unit Vector in the Equinoctical Coordinate Frame
F	Eccentric Longtitude
g_0	Surface Gravity Acceleration
$\underline{g}$	Unit Vector in the Equinoctial Coordinate Frame
G	Matrix of partial derivatives of Equinoctial elements with Respect to Velocity Vector
h	Equinoctial element
$\underline{h}$	Measurement Vector Function
H	$\frac{\partial}{\partial \underline{x}} \underline{h}$
i	Inclination
I	Identity Matrix
I_s	Specific Impulse
$I_{(\delta x_f)_i}$	Indicator Function for the Target Set
$I_{\underline{u}_m(t_j)}$	Indicator Function for the Control
J_0	Cost Criterion for the minimum Time Nominal Trajectory

J_1	Probability Function defined in (3.D.1)
J_2	Exponential Cost Criterion
k	Equinoctial Element
K	Kalmon Gain in the LEG Problem
K_i	Kalmon Gain in the extended Kalman Filter
L_j	Weighting Matrix for the Control
m	Mean angular motion
M_0	Mean Anomaly at the Epoch
$M(0)$	Covariance Matrix of the Initial Equinoctial Elements
$M'(0)$	Covariance Matrix of the Initial Position and Velocity
n_1, n_2, n_3	Components of $\underline{n}/\lvert y\rvert$ defined in Figure 2.3
$\underline{n}$	Random Thrust Error Vector
$\underline{n}_0$	Approximation of $\underline{n}$
n_{0j}	Discretized $\underline{n}_0$
N_1, N_2, N_3	Parameters representing the Strength of n_1, n_2, and n_3 respectively
N	Matrix representing the Strength of $\underline{n}$
N_0	Matrix representing the Strength of $\underline{n}_0$
N_{0j}	Matrix representing the Strength of $\underline{n}_{0j}$
p	Equinoctial Element
P	Estimation Error Covariance Matrix in the LEG Problem
$P_i, P_{\delta \underline{x}_i}$	Estimation Error Covariance Matrix in the extended Kalman Filter
q	Equinoctial Element
Q	Matrix Function defined in (3.E.13) and (3.E.14)
Q_f	Weighting Matrix for the Termainal State
r	$\lvert \underline{r} \rvert$

$\underline{r}$	Position Vector
R_1	Rotation Matrix defined
R_2	Rotation Matrix defined in (2.D.9)
$\underline{s}$	Unit Vector from Vehicle to Star
t	Time
t_f	Nominal Final Time
u_m	Scalar Function representing maximum Thrust Acceleration
$\underline{u}$	Desired Control
$\underline{u}_0$	Nominal Control
$\underline{u}'$	Vector which gives the direction of the desired Control
$\delta\underline{u}$	$\underline{u} - \underline{u}_o$
$\underline{v}$	Velocity Vector
$\underline{v}_m$	Measurement Noise Vector
V_i	Matrix representing the Strength of the Measurement Noise
w_x, w_y, w_z	Components of $\underline{w}$
$\underline{w}$	Unit Vector Normal to the Orbital Plane
$\underline{x}$	State Vector
$\hat{\underline{x}}$	Estimate of $\underline{x}$
$\tilde{\underline{x}}$	$\underline{x} - \underline{b}$
$\hat{\tilde{\underline{x}}}$	Estimate of $\tilde{\underline{x}}$
$\underline{x}_i$	Nominal Trajectory defined for the extended Kalman Filter
$\hat{\underline{x}}_i$	Estimate of $\underline{x}_i$
$\delta\underline{x}_i$	$\underline{x} - \underline{x}_i$
$\underline{x}_0$	Nominal Trajectory

$\underline{\bar{x}}(0)$	Expectation of $\underline{x}(0)$
$\delta\underline{x}$	$\underline{x} - \underline{x}_0$
$\underline{x}_f$	Target set Parameter defined in (2.E.2)
$\delta\underline{x}_f$	Target set Parameter defined in (2.E.2)
X_1	Position Coordinate relative to the Orbital Frame $\underline{f},\underline{g},\underline{w}$
$\dot{X}_1$	Velocity Coordinate relative to the Orbital Frame $\underline{f},\underline{g},\underline{w}$
X_{target}	Target Set
$\underline{y}$	Measurement Residual
Y	Covariance Matrix of the Measurement Residual
Y_1	Position Coordinate relative to the Orbital Frame $\underline{f},\underline{g},\underline{w}$
$\dot{Y}_1$	Velocity Coordinate relative to the Orbital Frame $\underline{f},\underline{g},\underline{w}$
$\underline{z}$	Measurement Vector
$\delta\underline{z}_i$	Defined in (4.A.4)
α	Angle defined in Figure 2.2
Λ	Feedback Control Gain Matrix
β	Auxiliary Variable defined in (2.A.19)
γ	Constant defined in (3.E.4)
$\delta(t-\tau)$	Dirac delta Function
δk_j	Kronecker delta
ε	Initial Thrust Acceleration in terms of g_o's
λ_0	Equinoctial Element
μ	Graviational Constant
$(\sigma_r)_1$	Standard Deviation of Position in Altitude Direction
$(\sigma_r)_2$	Standard Deviation of Position in Down Range Direction

$(\sigma_r)_3$	Standard Deviation of Position in Cross Track Direction
$(\sigma_v)_1$	Standard Diviation of Velocity in Altitude Direction
$(\sigma_v)_2$	Standard Deviation of Velocity in Down Range Direction
$(\sigma_v)_3$	Standard Diviation of Velocity in Cross Track Direction
τ	Time
Φ	State Transition Matrix
ψ	Matrix Defined in (3,C,21)
ω	Arguement of Perigee
Ω	Longtitude of the Ascending Node

CHAPTER I

INTRODUCTION

Recently, solar electric spacecraft propulsion systems of high efficiency have been developed. The advancements of this new technology have opened the road to a new era of space exploration and scientific research. Many space missions utilizing this propulsion system have been planned by the National Aeronautics and Space Administration for the second half of this decade. One of these missions utilizes the solar electric propulsion stage (SEPS) for the delivery and return of scientific payloads between near earth orbits and the geosynchronous orbits.

The purpose of this research is to develop a practical and efficient midcourse guidance and navigation system for these continuous low thrust vehicles. The vehicle is required to reach an allowable region near the desired geosynchronous orbit in a minimum amount of time. During this midcourse phase the angular position of the vehicle in the orbit is assumed to be unimportant. The magnitude of the thrust acceleration of the SEPS is constrainted to be bounded. The uncertainties due to the random initial state, the random thrusting error and sensor error are included.

In Chapter II a set of state variables is selected and a mathematical model is constructed. The motion of the vehicle is described by a nonlinear stochastic differential equation and uncertainties are modelled by stochastic processes. In Chapter III a minimum time nominal trajectory is defined and the equation of motion and measurement equation are linearized about this nominal trajectory. A meaningful cost criterion is constructed and a linear feedback control law is derived for the guidance system. In Chapter IV a navigation system is constructed and the complete closed-loop system is discussed. The computer simulation results of this system are presented and discussed in Chapter V. Finally conclusions are presented in Chapter VI and the various equations related to the state variables are presented in the Appendix.

CHAPTER II

THE MATHEMATICAL MODEL

The construction of a mathematical model is the most important step in the design of a gudiance and navigation system for continuous low thrust vehicles. In this chapter an appropriate set of state variables is selected. Then the equations describing the motion of the vehicle and the dynamics of the sensors are chosen. The uncertainties due to the random initial state, the random thrusting error and the sensor error are modelled statistically. Finally the problem considered in this research is stated mathematically.

2.A State Variables

The state variables used in this research are the equinoctial elements [4]. The most important advantage of these elements is that their equation of motion are free from singularities for zero eccentricity and zero inclination. This is not the case for the classical orbit elements [2].

The equinoctial elements can be defined in terms of the classical orbit elements as follows

$$\underline{x} = \begin{bmatrix} a \\ h \\ k \\ \lambda_0 \\ p \\ q \end{bmatrix} = \begin{bmatrix} a \\ e\ \sin(\omega+\Omega) \\ e\ \cos(\omega+\Omega) \\ M_0+\omega+\Omega \\ \tan(\frac{i}{2})\sin\Omega \\ \tan(\frac{i}{2})\cos\Omega \end{bmatrix} \qquad (2.A.1)$$

where a, e, i, M_0, ω and Ω are the classical orbit elements

a = semimajor axis

e = eccentricity

i = inclination

M_0 = mean anomaly at the epoch

ω = argument of perigee

Ω = longtitude of the ascending node .

Alternatively, the equinoctial elements can be defined in terms of the postition and velocity vectors. A coordinate system is defined for this purpose as shown in Figure 2.1. The unit vector normal to the orbital plane is given by

$$\underline{w} = \frac{\underline{r} \times \underline{v}}{|\underline{r} \times \underline{v}|} \qquad (2.A.2)$$

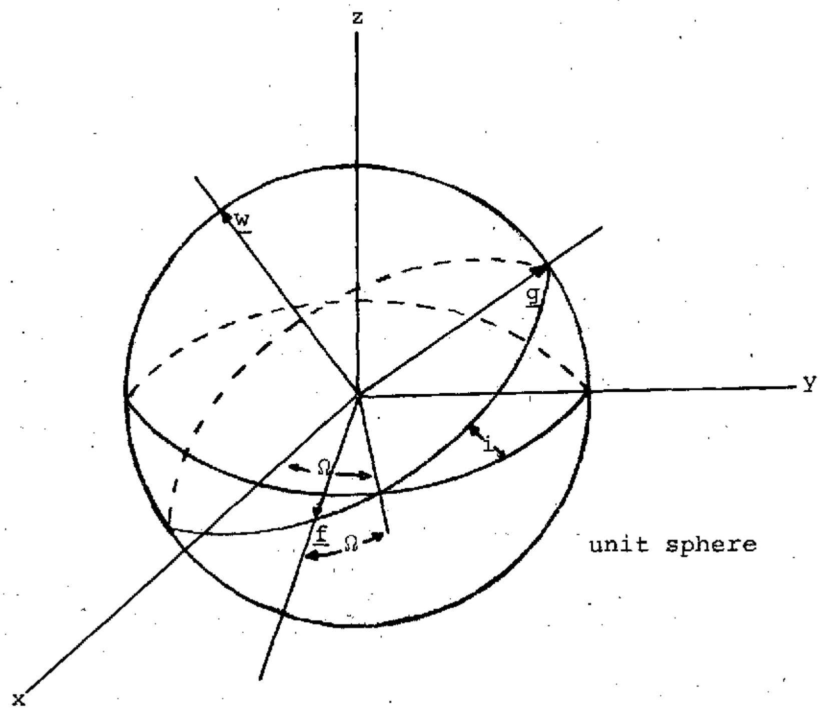
where $\underline{r}$ and $\underline{v}$ are the position and velocity vectors respectively. The components of this vector can be written in terms of the classical orbit elements as

$$\underline{w} = R_1 [0 \;\; 0 \;\; 1]^T$$

$$= \begin{bmatrix} \sin \Omega \sin i \\ -\cos \Omega \sin i \\ \cos i \end{bmatrix} \qquad (2.A.3)$$

where R_1 is the rotation matrix

$$R_1 = \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.A.4)$$



Equinoctial Coordinate Frame

Figure 2.1

Using (2.A.3) in (2.A.1) the equinoctial elements p and q can be written in terms of these components

$$p = \frac{\omega_x}{1 + \omega_z} \tag{2.A.5}$$

$$q = \frac{-\omega_y}{1 + \omega_z} \tag{2.A.6}$$

The unit vectors $\underline{f}$ and $\underline{g}$ defined in Figure 2.1 can now be written in terms of p and q

$$\underline{f} = R_1 \, [1 \quad 0 \quad 0]^T$$

$$= \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 1 - p^2 + q^2 \\ 2\, p\, q \\ -2\, p \end{bmatrix} \tag{2.A.7}$$

$$\underline{g} = R_1 \, [\, 0 \quad 1 \quad 0]^T$$

$$= \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 2\, p\, q \\ 1 + p^2 - q^2 \\ 2\, q \end{bmatrix} \tag{2.A.8}$$

The elements h and k are seen to be the components of the eccentricity vector in the directions of these unit vectors and are given by

$$h = \underline{e}^T \, \underline{g} \tag{2.A.9}$$

$$k = \underline{e}^T \, \underline{f} \tag{2.A.10}$$

where $\underline{e}$ is the eccentricity vector

$$\underline{e} = - \frac{\underline{r}}{|\underline{r}|} - \frac{(\underline{r} \times \underline{v}) \times \underline{v}}{\mu} \tag{2.A.11}$$

The element a is given by

$$a = \left(\frac{2}{|\underline{r}|} - \frac{|\underline{v}|^2}{\mu}\right)^{-1} \tag{2.A.12}$$

If the components of the position and velocity vectors along the unit vecotrs f and g are denoted by

$$x_1 = \underline{r}^T \underline{f} \tag{2.A.13}$$

$$y_1 = \underline{r}^T \underline{g} \tag{2.A.14}$$

$$\dot{x}_1 = \underline{v}^T \underline{f} \tag{2.A.15}$$

$$\dot{y}_1 = \underline{v}^T \underline{g} \tag{2.A.16}$$

the eccentric longtitude F can be written as

$$\cos F = k + \frac{(1-k^2\beta)X_1 - h\ k\ \beta Y_1}{a\sqrt{1-h^2-k^2}} \tag{2.A.17}$$

$$\sin F = h + \frac{(1-h^2\beta)Y_1 - h\ k\ \beta X_1}{a\sqrt{1-h^2-k}} \tag{2.A.18}$$

where

$$\beta = \frac{1}{1 + \sqrt{1-h^2-k^2}} \tag{2.A.19}$$

The remaining element λ_0 is given by Kepler's equation

$$\lambda_0 = F - k \sin F + h \cos F - \sqrt{\frac{\mu}{a^3}}\, t \tag{2.A.20}$$

where t is the time measured from the epoch.

These equations are the transformations from classical orbit elements or position and velocity vectors to equinoctial elements. The inverse transformation from equinoctial elements to position and velocity is also included here for convenience.

To calculate the position and velocity vectors from equinoctial elements, Kepler's equation (2.A.20) must be solved for the eccentricity longtitude F. Then the position and velocity vectors are given by

$$\underline{r} = X_1 \underline{f} + Y_1 \underline{g}$$

$$\underline{v} = \dot{X}_1 \underline{f} + \dot{Y}_1 \underline{g}$$

$$\dot{X}_1 = a[1-h^2\beta)\cos F + h\ k\ \beta\ \sin F - k] \tag{2.A.23}$$

$$\dot{Y}_1 = a[1-k^2\beta)\sin F + h\ k\ \beta\ \cos F - h] \tag{2.A.24}$$

$$\dot{X}_1 = \frac{\sqrt{\mu a}}{r}[h\ k\ \beta\ \cos F - (1-h^2\beta)\ \sin F] \tag{2.A.25}$$

$$\dot{Y}_1 = \frac{\sqrt{\mu a}}{r}[(1-k^2\beta)\cos F - h\ k\ \beta\ \sin F] \tag{2.A.26}$$

where

$$r = a[1-k\ \cos F - h\ \sin F] \tag{2.A.27}$$

2.B Equation of Motion

The only forces assumed to be acting on the vehicle are the inverse square gravitational attraction of earth, the desired engine thrust and the random thrusting error. The motion of the vehicle is described by a nonlinear stochastic differential equation [10].

$$\frac{d\underline{x}}{dt} = G(\underline{x},t)\underline{u} + G(\underline{x},t)\underline{n} \tag{2.B.1}$$

where $\underline{u}$ is the desired engine thrust, $\underline{n}$ is the random thrusting error and $G(\underline{x},t)$ is a 6x3 matrix of the partial derivatives of the equinoctial elements with respect to the velocity vector. The $G(\underline{x},t)$ matrix is given by

$$G(\underline{x},t) = \frac{\partial \underline{x}}{\partial \underline{v}} \tag{2.B.2}$$

where

$$\frac{\partial \bar{a}}{\partial \underline{v}} = \frac{2a^2}{\mu} \underline{v}^T \tag{2.B.3}$$

$$\frac{\partial h}{\partial \underline{v}} = \sqrt{\frac{1-h^2-k^2}{a\mu}} \left[\left(\frac{\partial X_1}{\partial k} - h\beta \frac{\dot{X}_1}{m}\right) f + \left(\frac{\partial Y_1}{\partial k} - h\beta \frac{\dot{Y}_1}{m}\right) g\right]^T$$

$$+ \frac{k(qY_1 - pX_1)}{ma^2\sqrt{1-h^2-k^2}} \underline{w}^T \tag{2.B.4}$$

$$\frac{\partial k}{\partial \underline{v}} = -\sqrt{\frac{1-h^2-k^2}{a\mu}} \left[\left(\frac{\partial X_1}{\partial h} + k\beta \frac{\dot{X}_1}{m}\right) f + \left(\frac{\partial Y_1}{\partial h} + k\beta \frac{\dot{Y}_1}{m}\right) g\right]^T$$

$$- \frac{h(qY_1 - pX_1)}{ma^2\sqrt{1-h^2-k^2}} \underline{w}^T \tag{2.B.5}$$

$$\frac{\partial \lambda_0}{\partial \underline{v}} = - \frac{2}{ma^2} \left[\underline{r} - \frac{3}{2} \underline{v}\, t\right]^T + \sqrt{\frac{1-h^2-k^2}{a\mu}} \beta\left[\left(h \frac{\partial X_1}{\partial h}\right.\right.$$

$$\left.\left. + k\frac{\partial X_1}{\partial k}\right) \underline{f} + \left(h \frac{\partial Y_1}{\partial h} + k \frac{\partial Y_1}{\partial k}\right) \underline{g}\right]^T$$

$$+ \frac{(qY_1-pX_1)}{ma^2\sqrt{1-h^2-k^2}} \underline{w}^T \tag{2.B.6}$$

$$\frac{\partial p}{\partial \underline{v}} = \frac{1+p^2+q^2}{2ma^2\sqrt{1-h^2-k^2}} Y_1 \underline{w}^T \tag{2.B.7}$$

$$\frac{\partial q}{\partial \underline{v}} = \frac{1+p^2+q^2}{2ma^2\sqrt{1-h^2-k^2}} X_1 \underline{w}^T \tag{2.B.8}$$

2.C Measurement Equation

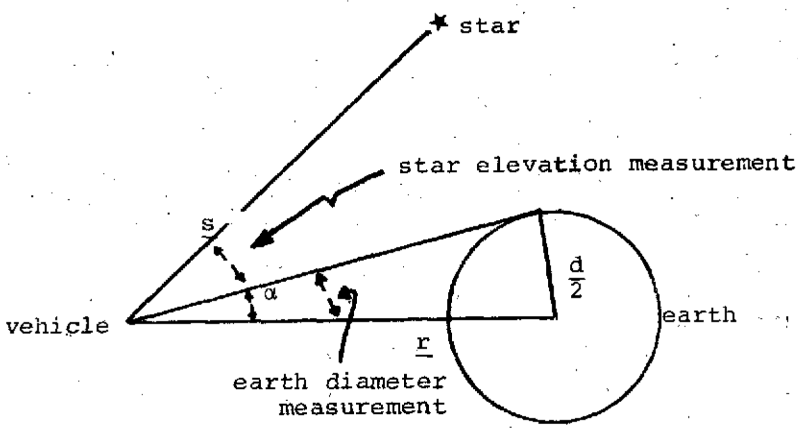
Sensors are used to make measurements and update the vehicle's estimate of it's state. These measurements which are corrupted by random sensor errors are assumed to be made at discrete instants of time. The types and the schedule of these measurements are assumed to be fixed. The measurement equation is given by

$$\underline{z}(t_i) = \underline{h}(\underline{x}, t_i) + \underline{v}_m(t_i) \tag{2.C.1}$$

$$0 = t_i < t_2 < \ldots < t_m = t_f$$

where $\underline{z}(t_i)$ is the measurement vector, $\underline{h}(\underline{x},t_i)$ is a vector function of the state and $\underline{v}_m(t_i)$ is the vector of the random sensor error. The form of the vector function $\underline{h}(\underline{x},t_i)$ depends on the type of measurement. For example, if a earth-diameter and a star-elevation measurement are taken simultaneously, the vector function $\underline{h}(\underline{x},t_i)$ given by

$$\underline{h}(\underline{x},t_i) = \begin{bmatrix} 2\sin^{-1}\left(\frac{d}{2r}\right) \\ \cos^{-1}\left(-\frac{\underline{s}\cdot\underline{r}}{r}\right) - \alpha \end{bmatrix} \tag{2.C.2}$$



Geometry of Earth-diameter
and Star-elevation Measurements

Figure 2.2

where $\underline{s}$ is the unit vector from the vehicle to the star and d is the diameter of earth. These measurements are pictured in Figure 2.2.

2.D Statistical Modelling of the Uncertainties

There are three sources of uncertainties considered. They are the random initial state, the random thrusting error and the sensor error. The initial state $\underline{x}(0)$ is assumed to be a Gaussian random vector. The mean and covariance of this vector is denoted by

$$E[\underline{x}(0)] = \bar{\underline{x}} \qquad (2.D.2)$$

$$E\{[\underline{x}(0) - \bar{\underline{x}}(0)][\underline{x}(0)-\bar{\underline{x}}(0)]^T\}= M(0) \qquad (2.D.2)$$

If the covariance matrix of the initial state is given in terms of the position and velocity vectors, the necessary transformation to the equinoctial elements is given by

$$M(0) = \begin{bmatrix} \frac{\partial \underline{x}}{\partial \underline{r}} & \frac{\partial \underline{x}}{\partial \underline{v}} \end{bmatrix}_{\bar{\underline{x}}(0)} M'(0) \begin{bmatrix} \frac{\partial \underline{x}}{\partial \underline{r}} & \frac{\partial \underline{x}}{\partial \underline{v}} \end{bmatrix}^T_{\bar{\underline{x}}(0)}$$

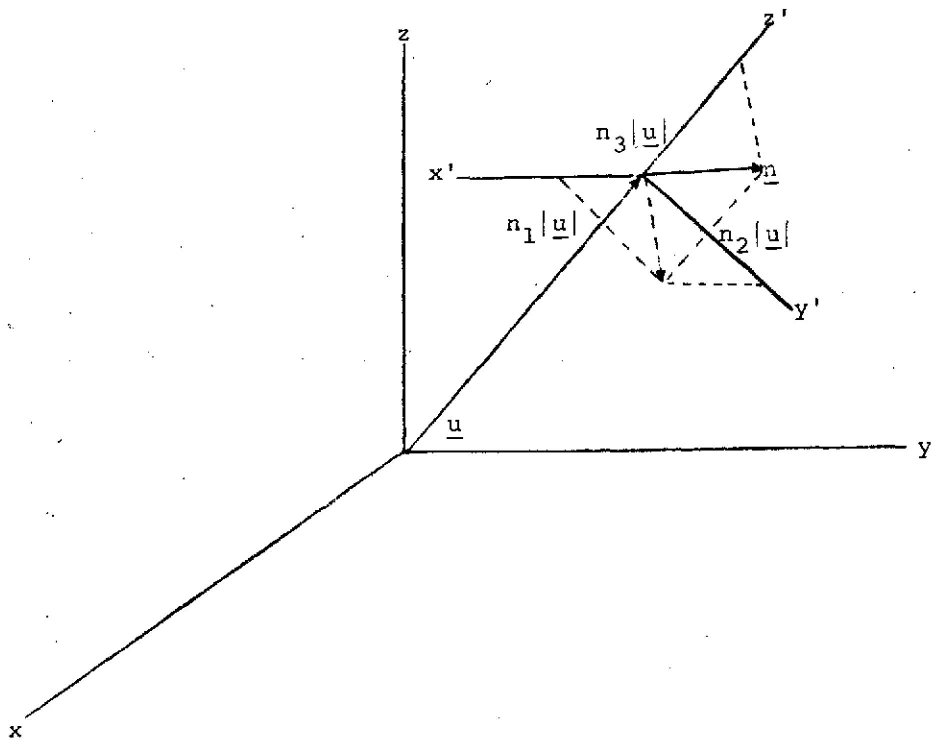
where $M'(0)$ is the covariance matrix of the initial position and velocity vectors. The matrix $\partial\underline{x}/\partial\underline{r}$ is included in the Appendix.

The random thrusting error is modelled by a zero mean white Gaussian random process

$$E[\underline{n}(t)]=0 \qquad (2.D.4)$$

$$E[\underline{n}(t)\underline{n}^T(\tau)] = N \ \delta(t-\tau) \qquad (2.D.5)$$

The matrix N, representing the strength of the process noise, is dependent on the desired engine thrust $\underline{u}$. Both the n and u vectors are pictured in Figure 2.3. The z' axis is defined in the direction of the vector $\underline{u}$. The x' and y' axes are defined in a plane normal to $\underline{u}$ to form a triad. The quantities n_1, n_2 and n_3 are assumed to be zero mean independent random processes



Process Noise Vector

Figure 2.3

$$E[n_i(t)] = 0 \qquad i=1,2,3 \tag{2.D.6}$$

$$E[n_i(t)\ n_i^T(\tau)] = N_i\ \delta(t-\tau) \qquad i=1,2,3 \tag{2.D.7}$$

$$E[n_i(t)\ n_j^T(\tau)] = 0 \qquad i\neq j \tag{2.D.8}$$

These quantities are also assumed to be independent of the vector $\underline{u}$. Let R_2 be the orthogonal transformation such that

$$\frac{\underline{u}}{|\underline{u}|} = R_2[\ 0 \quad 0 \quad 1]^T \tag{2.D.9}$$

Then the vector $\underline{n}(t)$ and it's correlation can be written as

$$\underline{n} = |\underline{u}|\ R_2[n_1\ n_2\ n_3]^T \tag{2.D.10}$$

$$E[\underline{n}(t)\underline{n}^T(\tau)] = E\left\{|\underline{u}|^2 R_2 \begin{bmatrix} n_1(t)n_1(\tau) & n_1(t)n_2(\tau) & n_1(t)n_3(\tau) \\ n_2(t)n_1(\tau) & n_2(t)n_2(\tau) & n_2(t)n_3(\tau) \\ n_3(t)n_1(\tau) & n_3(t)n_2(\tau) & n_3(t)n_3(\tau) \end{bmatrix} R_2^T \right\} \tag{2.D.11}$$

In view of (2.D.7) and (2.D.8), the correlation of $\underline{n}$ becomes

$$E[\underline{n}(t)\underline{n}^T(\tau)] = |\underline{u}|^2\ R_2 \begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix} R_2^T\ \delta(t-\tau) \tag{2.D.12}$$

Furthermore since n_1 and n_2 are defined in a plane normal to $\underline{u}$, it is reasonable to assume that

$$N_1 = N_2 \tag{2.D.13}$$

Using (2.D.5), (2.D.12) and (2.D.13) the matrix N is given by

$$N = N_1[I-\underline{u}\ \underline{u}^T] + N_3\ \underline{u}\ \underline{u}^T \tag{2.D.14}$$

The last source of uncertainties is the additive random sensor error $\underline{v}_m(t_i)$ in (2.C.1). This random sensor error is modelled as a zero mean white Gaussian random sequence

$$E[\underline{v}_m(t_i)] = 0 \tag{2.D.15}$$

$$E[\underline{v}_m(t_i)\underline{v}_m^T(t_j)] = V_i\delta_{ij} \tag{2.D.16}$$

Finally, the initial state $\underline{x}(0)$, the thrusting error $\underline{n}(t)$ and the sensor error $\underline{v}_m(t_i)$ are assumed to be independent of each other.

2.E Statement of the Problem

Given the nonlinear stochastic system in (2.B.1) and (2.C.1), the problem is to determine the engine thrust $\underline{u}(t)$, $t\leq t<t_f$ subject to the bounded magnitude constraint

$$|\underline{u}(t)| \leq u_m(t) \qquad 0\leq t<t_f \tag{2.E.1}$$

such that the vehicle will reach an allowable orbit near the desired geosynchronous orbit in a minimum amount of time. The function $u_m(t)$ in (2.E.1) represents the maximum amount of thrust acceleration that the SEPS can deliver at time t. The desired geosynchronous orbit is defined by the vector $\underline{x}_f$ where $(x_f)_4$ is free. The quantity $(x_f)_4$ is free because the angular position of the vehicle in the orbit is assumed to be unimportant during the midcourse phase. The allowable orbit near the desired geosynchronous orbit is defined by the target set X_{target} in the state space

$$X_{target} = \{\underline{x}(t_f) : |x_i(t_f)-(x_f)_i|<(\delta x_f)_i, \quad i=1, i\neq 4\} \tag{2.E.2}$$

where $\delta \underline{x}_f$ represents the maximum allowable deviations. The mathematical modelling of the problem is now completed. For stochastic systems it cannot be assured that the constraints (2.E.1) and (2.E.2) are satisfied. Probability measures are introduced in the next chapter to overcome this difficulty.

CHAPTER III

THE GUIDANCE SYSTEM

The problem formulated in the last chapter is a nonlinear stochastic optium control problem. This class of problems in general has no known solutions. In this chapter approximations are introduced so that a practical solution of the problem can be found. First a minimum time nominal trajectory is defined. This nominal trajectory is the solution to the problem if the uncertainties are absent. Then the nonlinear stochastic system (2.B.1) and (2.C.1) is linearized about this nominal trajectory. The linearized equation of motion is also discretized for convenience. An exponential cost criterion is formulated for the discretized linear stochastic system so that the vehicle will reach the allowable region near the desired geosynchronous orbit with the magnitude of the control bounded. Finally the solution to the linear-exponential-gaussian (LEG) terminal cost problem is presented and a linear feed-back control law is obtained for the guidance system.

3.A Nominal Trajectory

Since the objective of the problem is to guide the vehicle so that it will reach the target set in a minimum amount of time, the natural choice of the nominal trajectory is the minimum time trajectory. The state equation of this trajectory is given by

$$\frac{d\underline{x}_0}{dt} = G(\underline{x}_0,t)\,\underline{u}_0 \qquad (3.A.1)$$

where $\underline{x}_0$ is the nominal state and $\underline{u}_0$ is the nominal control. The initial and the final conditions are

$$\underline{x}_0(0) = \overline{\underline{x}}(0) \qquad (3.A.2)$$

$$\underline{x}_0(t_f) = \underline{x}_f \qquad (3.A.3)$$

where $(x_f)_4$ is free. The nominal control constraint is

$$|\underline{u}_0(t)| \leq u_m(t) \qquad (3.A.4)$$

The cost criterion to be minimized is

$$J_0 = \int_0^{t_f} dt \qquad (3.A.5)$$

The solution of this problem will fix the nominal $(x_f)_4$ and t_f. This problem may be solved by various existing techniques such as the minimum principle of Pontryagin or differential dynamic programming. Note that the nominal control $\underline{u}_0(t)$ will in general stay on the constraint boundary for this minimum time control problem.

3.B Linearization

The equation of motion and the measurement equation may now be linearized about the nominal trajectory. Define $\delta\underline{x}$ and $\delta\underline{u}$ as the deviations of the state and the control respectively from the nominal values

$$\delta\underline{x} = \underline{x} - \underline{x}_0 \qquad (3.B.1)$$

$$\delta\underline{u} = \underline{u} - \underline{u}_0 \qquad (3.B.2)$$

A linear expansion of the equation of motion yields

$$\frac{d\underline{x}}{dt} = G(\underline{x}_0,t)\underline{u} + [\frac{\partial}{\partial \underline{x}} G(\underline{x},t)\underline{u}]_{\underline{x}_0,\underline{u}_0} \delta\underline{x} + [\frac{\partial}{\partial \underline{u}} G(\underline{x},t)\underline{u}]_{\underline{x}_0,\underline{u}_0} \delta\underline{u}$$

$$+ G(\underline{x}_0,t)\,\underline{n} \qquad (3.B.3)$$

Using (3.B.1) and (3.B.2) this equation may be rewritten as

$$\frac{dx}{dt} = A(\underline{x}_0,\underline{u}_0,t)\underline{x} + B(\underline{x}_0,t)\underline{u} + B(\underline{x}_0,t)\,\underline{n}$$

$$-A(\underline{x}_0,\underline{u}_0,t)\,\underline{x}_0 \qquad (3.B.4)$$

where

$$A(\underline{x}_0,\underline{u}_0,t) = [\frac{\partial}{\partial \underline{x}} G(\underline{x},t)\underline{u}]_{\underline{x}_0,\underline{u}_0} \qquad (3.B.5)$$

$$B(\underline{x}_0,t) = G(\underline{x}_0,t) \qquad (3.B.6)$$

The 6x6 matrix $A(\underline{x}_0,\underline{u}_0,t)$ can be calculated explicitly in terms of the equinoctial elements. This matrix is included in the Appendix. The process noise $\underline{n}$ in (3.B.4) still depends on the desired control $\underline{u}$. Another approximation is made here so that $\underline{n}$ is approximated by a zero mean white gaussian random process $\underline{n}_0$. The statistics of this process is given by

$$E[\underline{n}_0(t)] = 0 \qquad (3.B.7)$$

$$E[\underline{n}_0(t)\underline{n}_0^T(\tau)] = N_0\ \delta(t-\tau) \qquad (3.B.8)$$

$$N_0 = N_1[I-\underline{u}_0\underline{u}_0^T] + N_3\ \underline{u}_0\ \underline{u}_0^T \qquad (3.B.9)$$

Now the linearized equation of motion becomes

$$\frac{d\underline{x}}{dt} = A(\underline{x}_0,\underline{u}_0,t)\underline{x} + B(\underline{x}_0,t)\underline{u} + B(\underline{x}_0,t)n_0$$

$$-A(\underline{x}_0,\underline{u}_0,t)\ \underline{x}_0 \qquad (3.B.10)$$

Similarily, a linear expansion of the measurement equation (2.C.1) yields

$$\underline{z}(t_i) = \underline{h}(\underline{x}_0,t_i) + [\frac{\partial}{\partial\underline{x}}\underline{h}(\underline{x},t_i)]_{\underline{x}_0}\delta\underline{x} + \underline{v}_m(t_i) \qquad (3.B.11)$$

Using (3.B.1) the linearized measurement equation becomes

$$\underline{z}(t_i) = H(\underline{x}_0,t_i)\underline{x}(t_i) + \underline{v}_m(t_i) + \underline{h}(\underline{x}_0,t_i) - H(\underline{x}_0,t_i)\underline{x}_0(t_i)$$

$$(3.B.12)$$

where

$$H(\underline{x}_0,t_i) = [\frac{\partial}{\partial\underline{x}}\underline{h}(\underline{x},t_i)]_{\underline{x}_0} \qquad (3.B.13)$$

The matrix $H(\underline{x}_0,t_i)$ is a nonrandom matrix which depends on the type of measurement taken. For the earth-diameter and star-elevation measurements, this matrix is given by

$$H(\underline{x}_0 t_i) = \begin{bmatrix} -\dfrac{2d}{r^2\sqrt{4r^2-d^2}}\,\underline{r}^T \\ \dfrac{r^2\underline{s}^T-(\underline{s}^T\underline{r})\underline{r}^T}{r^2\sqrt{r^2-(\underline{s}^T\underline{r})^2}} + \dfrac{d\underline{r}^T}{r^2\sqrt{4r^2-d^2}} \end{bmatrix}_{\underline{x}_0} \begin{bmatrix} \dfrac{\partial\underline{r}}{\partial\underline{x}} \end{bmatrix}_{\underline{x}_0} \qquad (3.B.14)$$

where $\frac{\partial\underline{r}}{\partial\underline{x}}$ is a 3x6 matrix of the partial derivatives of the position vector with respect to the equinoctial elements. This matrix is included in the Appendix.

3.C Discretization

It is more convienient to solve the guidance problem if the state is expressed as the sum of a stochastic process $\underline{\tilde{x}}$ and a nonrandom vector function $\underline{b}$. Define $\underline{\tilde{x}}$ and $\underline{b}$ by the following equations

$$\frac{d\underline{\tilde{x}}}{dt} = A(\underline{x}_0,\underline{u}_0,t)\underline{\tilde{x}} + B(\underline{x}_0,t)\underline{u} + B(\underline{x}_0,t)\underline{n}_0 \tag{3.C.1}$$

$$\frac{d\underline{b}}{dt} = A(\underline{x}_0,\underline{u}_0,t)\underline{b} - A(\underline{x}_0,\underline{u}_0,t)\underline{x}_0 \tag{3.C.2}$$

Then the state $\underline{x}$ is given by

$$\underline{x} = \underline{\tilde{x}} + \underline{b} \tag{3.C.3}$$

If the boundary condition on $\underline{b}$ is defined at the nominal final time t_f by

$$\underline{b}(t_f) = \underline{x}_f \tag{3.C.4}$$

then $\underline{\tilde{x}}(t_f)$ represents the deviation at the target. The corresponding initial condition of $\underline{b}$ is given by

$$\underline{b}(0) = \Phi^{-1}(t_f,0) + \int_0^{t_f} \Phi(\tau,0)A(\underline{x}_0,\underline{u}_0,\tau)\underline{x}_0(\tau)\ d\tau \tag{3.C.5}$$

where $\Phi(\tau,0)$ is the state transition matrix satisfying the following equations

$$\frac{\partial\Phi(\tau,t)}{\partial\tau} = A(\underline{x}_0,\underline{u}_0\ \tau)\ \Phi(\tau,t) \tag{3.C.6}$$

$$\Phi(t,t) = I \tag{3.C.7}$$

Using (3.C.3) the initial condition for $\underline{\tilde{x}}$ is given by

$$\underline{\tilde{x}}(0) = \underline{x}(0) - \underline{b}(0) \tag{3.C.8}$$

Therefore $\underline{\tilde{x}}(0)$ is also a Gaussian random vector with mean and covariance given by

$$E[\underline{\tilde{x}}(0)] = \underline{\bar{x}}(0) - \underline{b}(0)$$
$$E\{[\underline{\tilde{x}}(0) - \underline{\bar{x}}(0) + \underline{b}(0)]\ [\underline{\tilde{x}}(0) - \underline{\bar{x}}(0) + \underline{b}(0)]^T\} = M(0) \quad (3.C.9)$$

The linearized measurement equation (3.B.12) may also be written in terms of $\underline{\tilde{x}}$ and $\underline{b}$

$$\underline{z}(t_i) = H(\underline{x}_0,t_i)\underline{\tilde{x}}(t_i) + \underline{v}_m(t_i) + \underline{h}(\underline{x}_0,t_i) + H(\underline{x}_0,t_i)[\underline{b}(t_i) - \underline{x}_0(t_i)] \quad (3.C.11)$$

Now (3.C.1) and (3.C.2) may be discretized. Let the time interval $[0,t_f]$ be partitioned into n equal subintervals

$$0 = t_1<t_2<\ldots<t_{n+1} = t_f \quad (3.C.12)$$

Then the discretized equations are

$$\underline{\tilde{x}}(t_{j+1}) = \Phi(t_{j+1},t_j)\underline{\tilde{x}}(t_j) + \int_{t_j}^{t_{j+1}} \Phi(t_{j+1},\tau)\ B(\underline{x}_0,\tau)\underline{u}(\tau)d\tau + \underline{n}_{0j} \quad (3.C.13)$$

$$\underline{b}(t_j) = \Phi^{-1}(t_f,t_j)\underline{b}(t_f) + \int_{t_j}^{t_f} \Phi(t_j,\tau)\ A(\underline{x}_0,\underline{u}_0,\tau)\underline{x}_0(\tau)d\tau \quad (3.C.14)$$

where $\underline{n}_{0j}$ is a zero mean white Gaussian random sequence

$$E\ [\underline{n}_{0j}] = 0 \quad (3.C.15)$$

$$E\left[\underline{n}_{0j}\underline{n}_{0k}^T\right] = N_{0j}\ \delta_{jk} \quad (3.C.16)$$

$$N_{0j} = \int_{tj}^{tj+1} \Phi(t_{j+1},\tau)\ B(\underline{x}_0,\tau)\ N_0(\tau)\ B^T(\underline{x}_0,\tau)\ \Phi^T(t_{j+1},\tau)d\tau \quad (3.C.17)$$

If the number n of subintervals is very large, the control $\underline{u}(t)$ and the function $u_m(t)$ can be approximated by step functions

$$\underline{u}(t) = \underline{u}(t_j) \qquad t_j \le t < t_{j+1} \tag{3.C.18}$$

$$u_m(t) = u_m(t_j) \qquad t_j \le t < t_{j+1} \tag{3.C.19}$$

Then the discretized equation (3.C.13) becomes

$$\underline{\tilde{x}}(t_{j+1}) = \Phi(t_{j+1},t_j)\underline{\tilde{x}}(t_j) + \psi(t_{j+1},t_j)\underline{u}(t_j) + \underline{n}_{0j} \tag{3.C.20}$$

where

$$\psi(t_{j+1},t_j) = \int_{t_j}^{t_{j+1}} \Phi(t_{j+1},\tau)\; B(\underline{x}_0,\tau)\,d\tau \tag{3.C.21}$$

The control constraint equation (2.E.1) also become

$$|\underline{u}(t_j)| \le u_m(t_j) \qquad j=1,2,\ldots n \tag{3.C.22}$$

3.D. Exponential Cost Criterion

In the absence of the uncertainties, the minimum time nominal trajectory defined in section 3.A is the solution to the guidance problem. When the vehicle is disturbed by random thrusting error, the trajectory of its true motion will deviate from the nominal trajectory. However, there is no reason to try to drive the vehicle back to the nominal trajectory. Instead, the control sequence $\underline{u}(t_j)$ is determined to take into account the uncertainties so that the vehicle will reach the target set at the nominal time t_f with the control sequence bounded. For the stochastic system (3.C.11) and (3.C.20) it cannot be assured that these objectives can be met. Therefore probability measures are used to evaluate the effectiveness of the control in acheiving these objectives and the probability that the vehicle reaches the target and the controls stay within the bounds is to be maximized.

$$J_1 = P_r \, [\underline{x}(t_f) \, \varepsilon \, X_{target} \text{ and } |\underline{u}(t_1)| \leq u_m(t_1) \text{ and} \ldots$$
$$\text{and } |\underline{u}(t_n)| \leq u_m(t_n)] \qquad (3.D.1)$$

This probability can be written as an expectation if the following indicator functions are defined

$$I_{(\delta x_f)_i}[x(t_f)_i] = \begin{cases} 1 & \text{if } |x_i(t_f) - (x_f)_i| \leq (\delta x_f)_i \\ 0 & \text{if } |x_i(t_f) - (x_f)_i| > (\delta x_f)_i \end{cases}$$
$$i = 1,6; \; i \neq 4 \qquad (3.D.2)$$

$$I_{u_m}[\underline{u}(t_j)] = \begin{cases} 1 & \text{if } |\underline{u}(t_j)| \leq u_m(t_j) \\ 0 & \text{if } |\underline{u}(t_j)| > u_m(t_j) \end{cases}$$
$$j = 1,2\ldots,n \qquad (3.D.3)$$

Then the probability J_1 in (3.D.1) is the expectation of the product of these indicator functions

$$J_1 = E \left\{ \prod_{\substack{i=1,6 \\ i\neq 4}} I_{\delta x_{fi}}[x_i(t_f)] \prod_{j=1,n} I_{u_m}[\underline{u}(t_j)] \right\} \qquad (3.D.4)$$

Application of dynamic programming to this problem will determine the optimal control. The character of the control is such that maximum thrust is utilized until the estimated state reaches the target state. In effect the estimated state is driven to the target as quickly as possible and the essential character of the minimum time solution is also present in the maximum probability solution.

Since the objective of the SEPS spacecraft is to reach the target set in minimum time, the control sequence $\underline{u}(t_j)$ should satisfy the equality in (3.C.22). There is no known general solution for the problem of determining the control sequence to maximize the expectation J_1 in (3.D.4) subject to the linear stochastic system (3.C.11) and (3.C.20). The product of these indicator functions will now be approximated by a exponential cost function.

$$J_1 \cong E\left[\exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{n} \underline{u}^T(t_j) L_j \underline{u}(t_j) + (\underline{x}(t_f)-\underline{x}_f)^T Q_f (\underline{x}(t_f)-\underline{x}_f)\right]\right\}\right] \quad (3.D.5)$$

The weighting matrices L_j and Q_f must be chosen so that each term of the exponential function in (3.D.5) approximate the corresponding indicator function in (3.D.4). The normalized second moments of the indicator functions $I_{(\delta x_f)_i}[x_i(t_f)]$ are

$$\frac{1}{2(\delta x_f)_i} \int_{(x_f)_i-(\delta x_f)_i}^{(x_f)_i+(\delta x_f)_i} [x_i(t_f)-(x_f)_i]^2 \, dx(t_f)_i = \frac{[(\delta x_f)_i]^2}{3}$$

$$i = 1,6; \quad i \neq 4 \quad (3.D.6)$$

The normalized second moments of the indicator functions $I_{u_m}[\underline{u}(t_j)]$ are

$$\frac{1}{\frac{4}{3}\pi[u_m(t_j)]^3} \iiint [u_k(t_j)]^2 \, d\,u_x(t_j) \, d\,u_y(t_j) \, d\,u_z(t_j) = \frac{[u_m(t_j)]^2}{5}$$

$$k = x,y,z$$
$$j = 1,2,\ldots n \quad (3.D.7)$$

where the integration is over a sphere with radius $u_m(t_j)$. Therefore if the weighting matrices L_j and Q_f are chosen as

$$L_j = \frac{5}{[u_m(t_j)]^2} I \quad (3.D.8)$$

$$
Q_f = \begin{bmatrix}
\frac{1}{[(\delta x_f)_1]^2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{[(\delta x_f)_2]^2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{[(\delta x_f)_3]^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{[(\delta x_f)_5]^2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{[(\delta x_f)_6]^2}
\end{bmatrix} \quad (3.D.9)
$$

the normalized second moments of the indicators are the same as the normalized second moments of the corresponding terms in the exponential cost function. The solution for the problem of the maximization of the expectation of this exponential cost criteria is presented in the next section.

3.E Solution of the LEG Terminal Cost Problem

The linear discrete stochastic system obtained in the last four sections is given by

$$
\underline{\tilde{x}}(t_{j+1}) = \Phi(t_{j+1},t_j)\underline{\tilde{x}}(t_j) + \psi(t_{j+1},t_j)\underline{u}(t_j) + \underline{n}_{0j}
$$

$$
j = 1,2,\ldots n \qquad (3.E.1)
$$

$$
\underline{z}(t_i) = H(\underline{x}_0,t_i)\ \underline{\tilde{x}}(t_i) + \underline{v}_m(t_i) + \underline{h}(\underline{x}_0,t_i)
$$

$$
+ H(\underline{x}_0,t_i)\ [\underline{b}(t_i) - \underline{x}_0(t_i)]
$$

$$
i = 1,2,\ldots m \qquad (3.E.2)
$$

The initial state $\underline{\tilde{x}}(t_1)$ is a Gaussian random vector, the process noise $\underline{n}_{0j}$ and the measurement noise $\underline{v}_m(t_i)$ are zero mean white Gaussian random sequences all of which are statistically independent. The measurement noise covariance and the control weighting matrices are positive definite matrices. The process noise covariance and the terminal state weighting matrices are positive semi-definite matrices. The problem of the determination of the control sequence to maximize the expectation of the exponential cost criteria in (3.D.5) is called the linear-exponential-gaussian (LEG) terminal cost problem. The maximization of the expectation in (3.D.5) is the same as the minimization of the following

$$J_2 = E\left\{\gamma\ \exp[\tfrac{\gamma}{2}\sum_{j=1}^{n} \underline{u}^T(t_j)L_j\,\underline{u}(t_j) + \tfrac{\gamma}{2}\quad \underline{\tilde{x}}^T(t_f)\ Q_f\ \underline{\tilde{x}}(t_f)]\right\} \tag{3.E.3}$$

with

$$\gamma = -1 \tag{3.E.4}$$

This problem is treated in a paper by Speyer, Deyst and Jacobson [9]. The controls are restricted to be Borel functions of the past measurement history. The key in solving this problem is to utilize the results of the Kalman-Bucy filter [5] and dynamic programming [6]. For the terminal cost problem formulated here, the separation theorm [11] holds. The optimal feedback control is a linear function of the current state estimate

$$\underline{u}(t_j) = -\Lambda(t_j)\ \hat{\underline{\tilde{x}}}(tj) \tag{3.E.5}$$

where $\hat{\underline{\tilde{x}}}(t_j)$ denotes the current minimum variance estimate of $\underline{\tilde{x}}(t_j)$. Under the conditions of this problem, the state estimate is the mean of $\underline{\tilde{x}}(t_j)$ conditioned on the past measurement history $[\underline{z}(t_1),\ \underline{z}(t_2)\ldots\underline{z}(t_i)]$, $t_i \le t_j$. At a measurement, this conditional mean is updated by

$$\hat{\underline{x}}(t_j^+) = \hat{x}(t_j^-) + K(t_j)\ \underline{y}(t_j) \qquad (3.E.6)$$

where $\hat{\underline{x}}(t_j^+)$ and $\hat{\underline{x}}(t_j^-)$ denote the estimate of $\hat{\underline{x}}(t_j)$ after and before the measurement respectively. The measurement residual $\underline{y}(t_j)$ is given by

$$\underline{y}(t_j) = \underline{z}(t_j) - \underline{h}(\underline{x}_0,t_j) - H(\underline{x}_0,t_j)\ [\underline{b}(t_j)-\underline{x}_0(t_j)]$$
$$- H(\underline{x}_0,t_j)\ \hat{\underline{x}}(t_j) \qquad (3.E.7)$$

The Kalman gain $K(t_j)$ is given by

$$K(t_j) = P(t_j^+)\ H^T(\underline{x}_0,t_j)\ V^{-1}(t_j) \qquad (3.E.8)$$

where $P(t_j)$ is covariance matrix of the estimation error conditioned on the past measurement history. This conditional covariance matrix is updated at a measurement by

$$P(t_j^+) = P(t_j^-) - P(t_j^-)\ H^T(\underline{x}_0,t_j)\ [H(\underline{x}_0,t_j)\ P(t_j^-)\ H^T(\underline{x}_0,t_j)$$
$$+ V(t_j)]^{-1}\ H(\underline{x}_0,t_j)\ P(t_j^-) \qquad (3.E.9)$$

where $P(t_j^+)$ and $P(t_j^-)$ denote $P(t_j)$ before and after the measurement respectively. Between two measurements, $\hat{\underline{x}}(t_j)$ and $P(t_j)$ propagate according to the following equations

$$\hat{\underline{x}}(t_{j+1}) = \Phi(t_{j+1},t_j)\ \hat{\underline{x}}(t_j) + \psi(t_{j+1},t_j)\ \underline{u}(t_j) \qquad (3.E.10)$$

$$P(t_{j+1}) = \Phi(t_{j+1},t_j)\ P(t_j)\ \Phi^T(t_{j+1},t_j) + N_{0j} \qquad (3.E.11)$$

The initial state estimate and the error covariance matrix are given by the apriori statistics of $\hat{\underline{x}}(t_1)$. While the separation theorm holds for this problem, the certainty equivalence principle [3] does not hold. The feedback control gain $\Lambda(t_j)$ depends on the noise characteristics. This dependence reflects the quality of the state estimate. The feedback control gain matrix is given by

$$\Lambda(t_j) = [L_j + \psi^T(t_{j+1},t_j)\ Q(t_{j+1})\psi(t_{j+1},t_j)]^{-1}\psi^T(t_{j+1},t_j)\ Q(t_{j+1})\ \Phi(t_{j+1},t_j) \tag{3.E.12}$$

The matrix $Q(t_j)$ is given by a backward difference equation

$$Q(t_{j-1}) = \Phi^T(t_j,t_{j-1}) \Big\{Q(t_j) - Q(t_j)\ \psi(t_j,t_{j-1})\ [\psi^T(t_j,t_{j-1})\ Q(t_j)\ \psi(t_j,t_{j-1}) + L_{j-1}]^{-1}\ \psi^T(t_j,t_{j-1})\ Q(t_j)\Big\}\ \Phi(t_j,t_{j-1}) \tag{3.E.13}$$

where at a measurement

$$Q(t_j^-) = Q(t_j^+) + YQ(t_j^+)\ K(t_j)\ [Y^{-1}(t_j) - YK^T(t_j)\ Q(t_j^+)\ K(t_j)]^{-1}\ K^T(t_j)\ Q(t_j^+) \tag{3.E.14}$$

Note that at a measurement, (3.E.12) becomes

$$\Lambda(t_j) = [L_j + \psi^T(t_{j+1},t_j)\ Q(t_{j+1}^-)\ \psi(t_{j+1},t_j 0]^{-1}\ \psi^T(t_{j+1},t_j)\ Q(t_{j+1}^-)\ \Phi(t_{j+1},t_j) \tag{3.E.15}$$

where $Q(t_{j+1})$ is replaced by $Q(t_{j+1}^-)$.

The matrix $Y(t_j)$ is the covariance matrix of the measurement residual

$$Y(t_j) = H(\underline{x}_0,t_j)\ P(t_j)\ H^T(\underline{x}_0,t_j) + V(t_j) \tag{3.E.16}$$

Equation (3.E.14) shows the dependance of the control gain matrix on the noise characteristics explicitly. The terminal condition of $Q(t_j)$ is given by

$$Q(t_{n+1}) = Q_f + \gamma Q_f \,[P^{-1}(t_f^{+}) - \gamma Q_f\,]^{-1}\, Q_f \qquad (3.E.17)$$

3.F The Guidance System

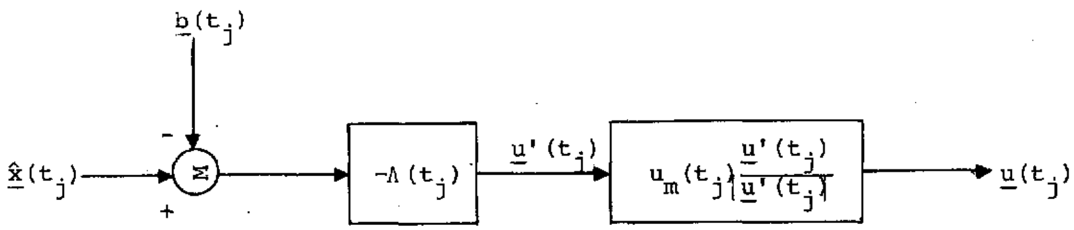
The linear feedback control law obtained in the solution of the LEG terminal cost problem can be used for the midcourse guidance of the SEPS Spacecraft. Since the overall objective of the SEPS Spacecraft is to reach the target set in minimum time, full thrust acceleration will be used to propel the vehicle. The LEG guidance law is used to determine thrust direction only and full thrust magnitude is always utilized. Therefore the guidance law is given by

$$\underline{u}(t_j) = \underline{u}_m(t_j)\,\frac{\underline{u}'(t_j)}{|\underline{u}'(t_j)|} \qquad (3.F.1)$$

where using (3.C.2) and (3.E.5) $\underline{u}'(t_j)$ is given by

$$\underline{u}'(t_j) = -\Lambda(t_j)\;[\hat{\underline{x}}(t_j) - \underline{b}(t_j)] \qquad (3.F.2)$$

The on-board guidance system is only required to perform the vector subtraction and matrix multiplication in (3.F.2). The feedback control gain matrix $\Lambda(t_j)$ and the vector $\underline{b}(t_j)$ can be computed before the mission and stored in a computer on the SEPS for real time mission usage. The navigation system to estimate the state $\hat{\underline{x}}(t_j)$ is discussed in the next chapter. The guidance law (3.F.2) is pictured in Figure 3.1.



The Guidance System

Figure 3.1

CHAPTER IV

THE NAVIGATION SYSTEM

In the last chapter a linear feedback control law was obtained for the guidance of the SEPS Spacecraft. Using this control law, the vehicle will fly in a trajectory neighboring the minimum time nominal trajectory. Therefore the linear Kalman filter presented in section 3.E is not adequate to estimate the vehicle's state. In this chapter the extended Kalman filter [7] which is adequate for neighbouring trajectory estimation is discussed. This estimator, together with the linear feedback control law obtained in the last chapter forms the complete closed-loop midcourse guidance and navigation system for the SEPS Spacecraft.

4.A Extended Kalman Filter

The extended Kalman filter has the same structure as the linear Kalman filter. However, instead of linearizing about the minimum time nominal trajectory alone, the extended Kalman filter is linearized about a number of nominal trajectories. After each measurement, a new estimate of the state is obtained. This new estimate is used to define a new nominal trajectory. Then the equation of motion and the measurement equation are linearized about this new nominal trajectory.

It is more convenient to discuss the extended Kalman filter if the continuous equation of motion (2.B.1) is used. Now suppose the control $\underline{u}(t)$, $0 \leq t < t_f$ is known. Let the estimate of the state and the error covariance matrix of this estimate after the measurement at time t_i be $\hat{\underline{x}}(t_i^+)$ and $P(t_i^+)$ respectively. This estimate is used to define a nominal trajectory $\underline{x}_i(t)$ by

$$\frac{d\underline{x}_i(t)}{dt} = G(\underline{x}_i, t)\ \underline{u}(t) \qquad t_i \le t \tag{4.A.1}$$

$$\underline{x}_i(t_i) = \hat{\underline{x}}(t_i^+) \tag{4.A.2}$$

The subscribt i is used to emphasize the dependence of the nominal trajectory $\underline{x}_i(t)$ on the state estimate $\underline{x}(t_i)$. Define $\delta\underline{x}_i(t)$ and $\delta\underline{z}_i(t_{i+1})$ by

$$\delta\underline{x}_i(t) = \underline{x}(t) - \underline{x}_i(t) \qquad t_i \le t \tag{4.A.3}$$

$$\delta\underline{z}_i(t_{i+1}) = \underline{z}(t_{i+1}) - \underline{h}(\underline{x}_i, t_{i+1}) \tag{4.A.4}$$

Linearization of (2.B.1) and (2.C.1) about this nominal trajectory yields

$$\frac{d\delta\underline{x}_i(t)}{dt} = A(\underline{x}_i, \underline{u}, t)\ \delta\underline{x}_i(t) + B(\underline{x}_i, t)\underline{n} \qquad t_i \le t \tag{4.A.5}$$

$$\delta\underline{z}_i(t_{i+1}) = H(\underline{x}_i, t_{i+1})\ \delta\underline{x}_i(t_{i+1}) + \underline{v}_m(t_{i+1}) \tag{4.A.6}$$

Now the linear filtering theory can be applied to estimate $\delta\underline{x}_i(t)$. Before the measurement at time t_{i+1}, the estimate $\delta\hat{\underline{x}}_i(t)$ and the error covariance matrix $P_{\delta\underline{x}_i}(t)$ of this vector are given by the following equations

$$\frac{d\delta\hat{\underline{x}}_i(t)}{dt} = A(\underline{x}_i, \underline{u}, t)\ \delta\hat{\underline{x}}_i(t) \tag{4.A.7}$$

$$\frac{dP_{\delta\underline{x}_i}(t)}{dt} = A(\underline{x}_i, \underline{u}, t)\ P_{\delta\underline{x}_i}(t) + P_{\delta\underline{x}_i}(t)\ A^T(\underline{x}_i, \underline{u}, t)$$

$$+ B(\underline{x}_i, t) N\ B^T(\underline{x}_i, t) \qquad t_i \le t < t_{i+1} \tag{4.A.8}$$

Using (4.A.3) this estimate is related to the estimate $\hat{\underline{x}}(t)$ of the state $\underline{x}(t)$ by

$$\delta\hat{\underline{x}}_i(t) = \hat{\underline{x}}(t) - \underline{x}_i(t) \qquad t_i \leq t \tag{4.A.9}$$

Also using (4.A.2)

$$\delta\hat{\underline{x}}_i(t_i^+) = 0 \tag{4.A.10}$$

and in view of (4.A.7)

$$\delta\hat{\underline{x}}_i(t) = 0 \qquad t_i \leq t < t_{i+1} \tag{4.A.11}$$

Therefore before the measurement at time t_{i+1}, the estimate of the state is given by the nominal trajectory $\underline{x}_i(t)$

$$\hat{\underline{x}}(t) = \underline{x}_i(t) \tag{4.A.12}$$

$$\frac{d\hat{\underline{x}}(t)}{dt} = G(\hat{\underline{x}},t)\ \underline{u}(t)$$

$$t_i \leq t < t_{i+1} \tag{4.A.13}$$

At the measurement at time t_{i+1}, $\delta\underline{x}(t)$ and $P_{\delta\underline{x}_i}(t)$ are updated by the following equations

$$\delta\hat{\underline{x}}_i(t_{i+1}^+) = \delta\hat{\underline{x}}_i(t_{i+1}^-) + K_i(t_{i+1})\ \{\delta\underline{z}_i(t_{i+1}) - H(\underline{x}_i,t_{i+1})\ \delta\hat{\underline{x}}_i(t_{i+1}^-)\} \tag{4.A.14}$$

$$K_i(t_{i+1}) = P_{\delta\underline{x}_i}(t_{i+1}^+)\ H^T(\underline{x}_i,t_{i+1})\ V^{-1}(t_{i+1}) \tag{4.A.15}$$

$$P_{\delta\underline{x}_i}(t_{i+1}^+) = P_{\delta\underline{x}_i}(t_{i+1}^-) - P_{\delta\underline{x}_i}(t_{i+1}^-)H^T(\underline{x}_i,t_{i+1})\ [H(\underline{x}_i,t_{i+1})\ P_{\delta\underline{x}_i}(t_{i+1}^-)H^T(\underline{x}_i,t_{i+1}) + V(t_{i+1})]^{-1}\ H(\underline{x}_i,t_{i+1})\ P_{\delta\underline{x}_i}(t_{i+1}^-) \tag{4.A.16}$$

Using (4.A.4), (4.A.9), (4.A.11) and (4.A.12), (4.A.14) becomes

$$\underline{\hat{x}}(t_{i+1}^{+}) = \underline{\hat{x}}(t_{i+1}^{-}) + K_i(t_{i+1}) \{z(t_{i+1}) - \underline{h}[\underline{\hat{x}}(t_{i+1}^{-}),t_{i+1}]\} \tag{4.A.17}$$

Using (4.A.3) the error covariance matrix $P_{\delta\underline{x}_i}(t)$ of the estimate of $\delta\underline{x}_i(t)$ and the error covariance matrix $P_i(t)$ of the estimate of $\underline{x}(t)$ are the same

$$\frac{dP_i(t)}{dt} = A(\underline{\hat{x}},\underline{u},t)\, P_i(t) + P_i(t)\, A^T(\underline{\hat{x}},\underline{u},t) + B(\underline{\hat{x}},t)N\, B^T(\underline{\hat{x}},t)$$

$$t_i \leq t < t_{i+1} \tag{4.A.18}$$

$$P_i(t_{i+1}^{+}) = P_i(t_{i+1}^{-}) - P_i(t_{i+1}^{-})\, H^T[\underline{\hat{x}}(t_{i+1}^{-}),t_{i+1}]\{H[\underline{\hat{x}}(t_{i+1}^{-}),t_{i+1}]\, P_i(t_{i+1}^{-})\, H^T[\underline{\hat{x}}(t_{i+1}^{-}),t_{i+1}] + V(t_{i+1})\}^{-1}\, H[\underline{\hat{x}}(t_{i+1}^{-}),t_{i+1}]\, P_i(t_{i+1}^{-}) \tag{4.A.19}$$

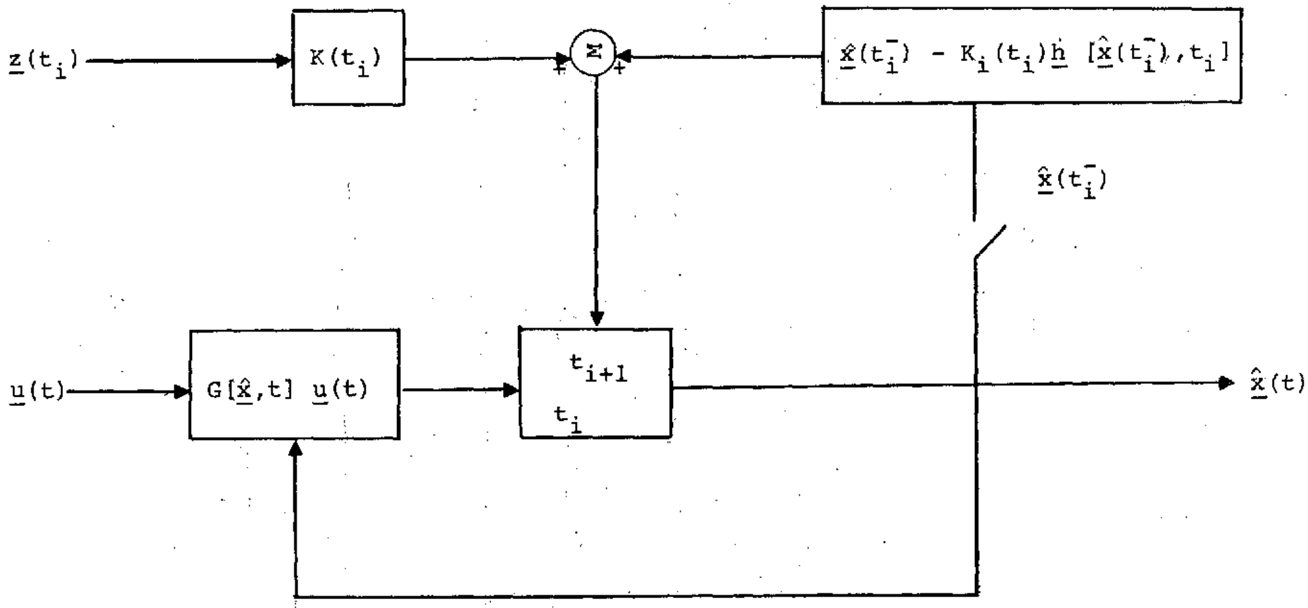
Hence (4.A.17) can be rewritten as

$$\underline{\hat{x}}(t_{i+1}^{+}) = \underline{\hat{x}}(t_{i+1}^{-}) + K_i(t_{i+1}) \{\underline{z}(t_{i+1}) - h[\underline{\hat{x}}(t_{i+1}^{-}),t_{i+1}]\} \tag{4.A.20}$$

where now the Kalman gain $K_i(t_{i+1})$ is given by

$$K_i(t_{i+1}) = P_i(t_{i+1})\, H^T[\underline{\hat{x}}(t_{i+1}^{-}),t_{i+1}]\, V^{-1}(t_{i+1}) \tag{4.A.21}$$

Now the new estimate $\underline{\hat{x}}(t_{i+1}^{+})$ can be used to define a new nominal trajectory similar to (4.A.1) and (4.A.2) and the preceding method can be repeated. The result is the extended Kalman filter given by (4.A.13), (4.A.18) to (4.A.21). This estimator is pictured in Figure 4.1.

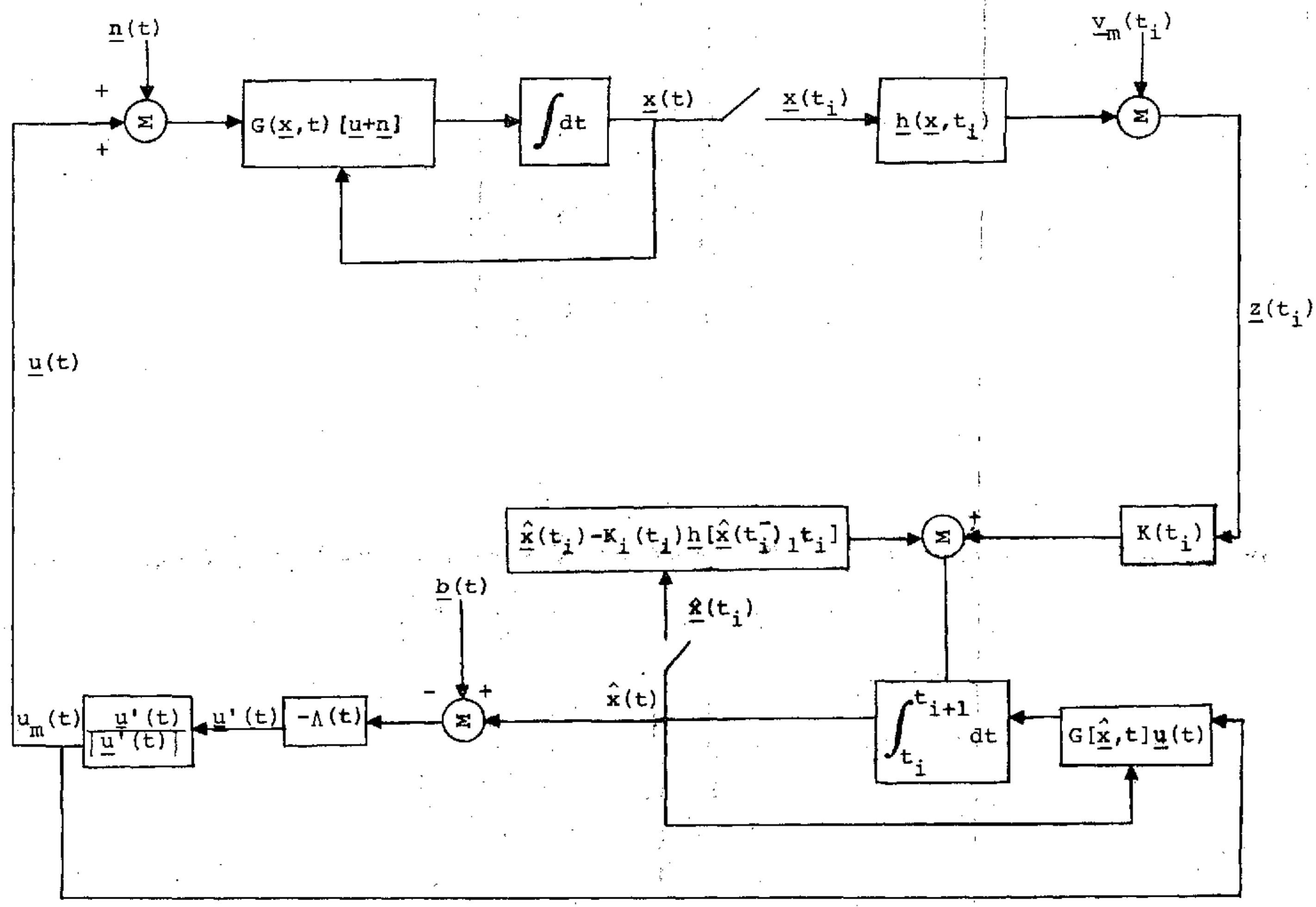


The Navigation System

Figure 4.1

4.B The Closed-loop Guidance and Navigation System

The linear feedback control law in Chapter 3 and the extended Kalman filter in the last section forms the complete closed-loop midcourse guidance and navigation system for the SEPS Spacecraft. The on-board guidance system consists of the linear feedback control law (3.F.1) and (3.F.2) where $\Lambda(t_j)$ and $\underline{b}(t_j)$ are precomputable quantities. Note that $\Lambda(t_j)$ is computed by using the equations in Chapter 3 where the quantities $P(t_j)$ and $K(t_j)$ are not the same as the quantities $P_i(t)$ and $K_i(t)$ in the last section. The quantities $P(t_j)$ and $K(t_j)$ are computed along the minimum time nominal trajectory while the quantities $P_i(t)$ and $K_i(t)$ along a number of nominal trajectories. This control law will guide the vehicle to fly along a trajectory neighbouring to the minimum time nominal trajectory and reach the target set at the nominal final time t_f. The thrust acceleration is always fully utilized to propel the vehicle and the control is always on the constraint boundary for this minimum time mission. The on-board navigation system consists of the extended Kalman filter (4.A.13) and (4.A.18) to (4.A.21) where all the quantities must be computed on-board the vehicle. The on-board computation of these quantities is the most important disadvantage of this navigation system. The closed-loop system is pictured in Figure 4.2. Although this guidance and navigation system is designed for the midcourse phase, it can also be used for the terminal phase by including the term for the angular position of the vehicle in the terminal state weighting matrix. However, in this case the objective of reaching the target set which now included the angular position of the vehicle is more difficult to meet than the midcourse case, unless the nominal mission time is very short.



The closed-loop System

Figure 4.2

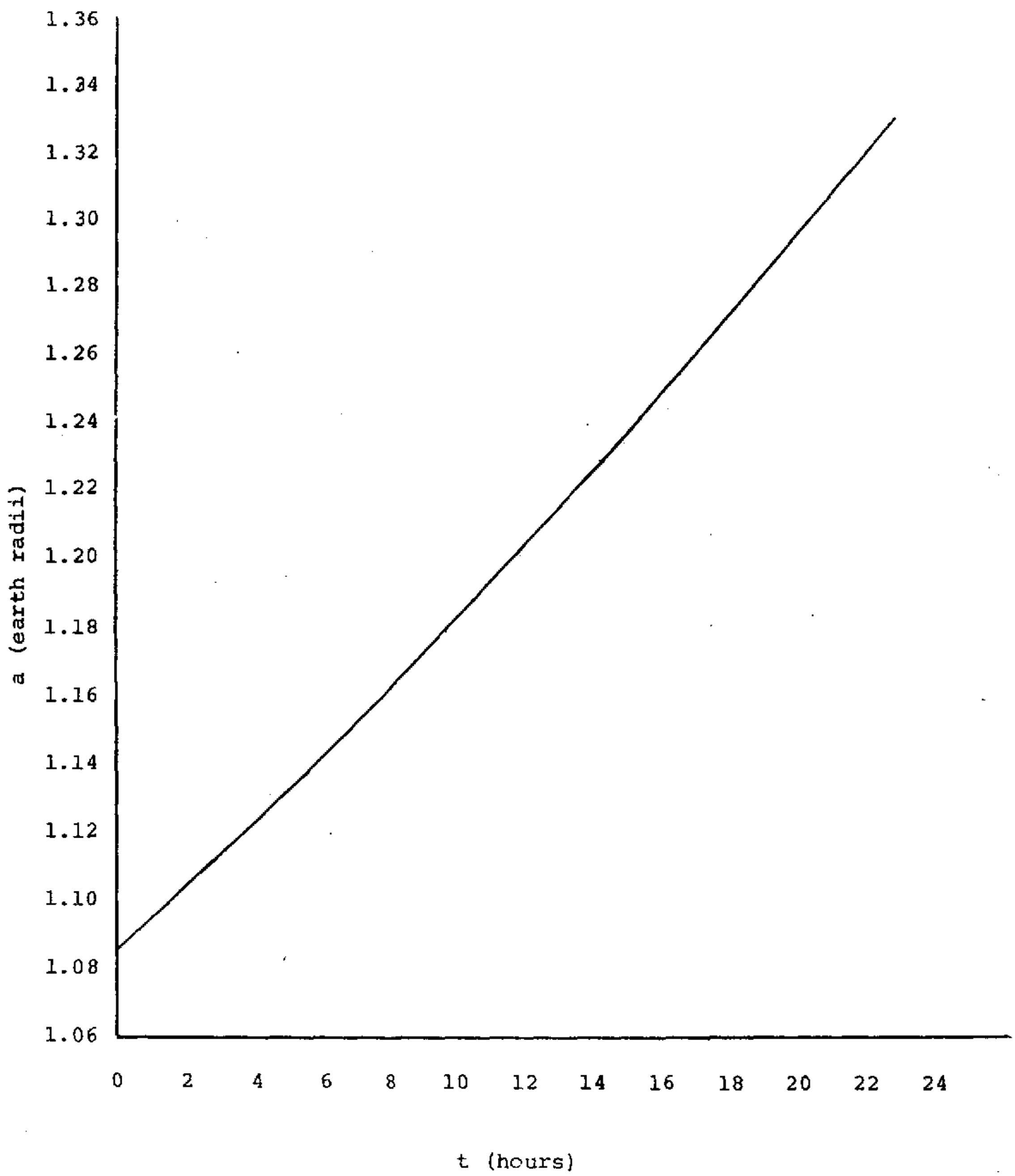
CHAPTER V

SIMULATION RESULTS AND DISCUSSION

A computer program has been prepared to simulate this midcourse guidance and navigation system in real time. Although the system was originally designed for missions from near earth orbits to geosynchronous orbits, the simulation of a shorter mission should equally well reveal the character and performance of the system. The simulation results of this short mission, together with a discussion are presented in this chapter.

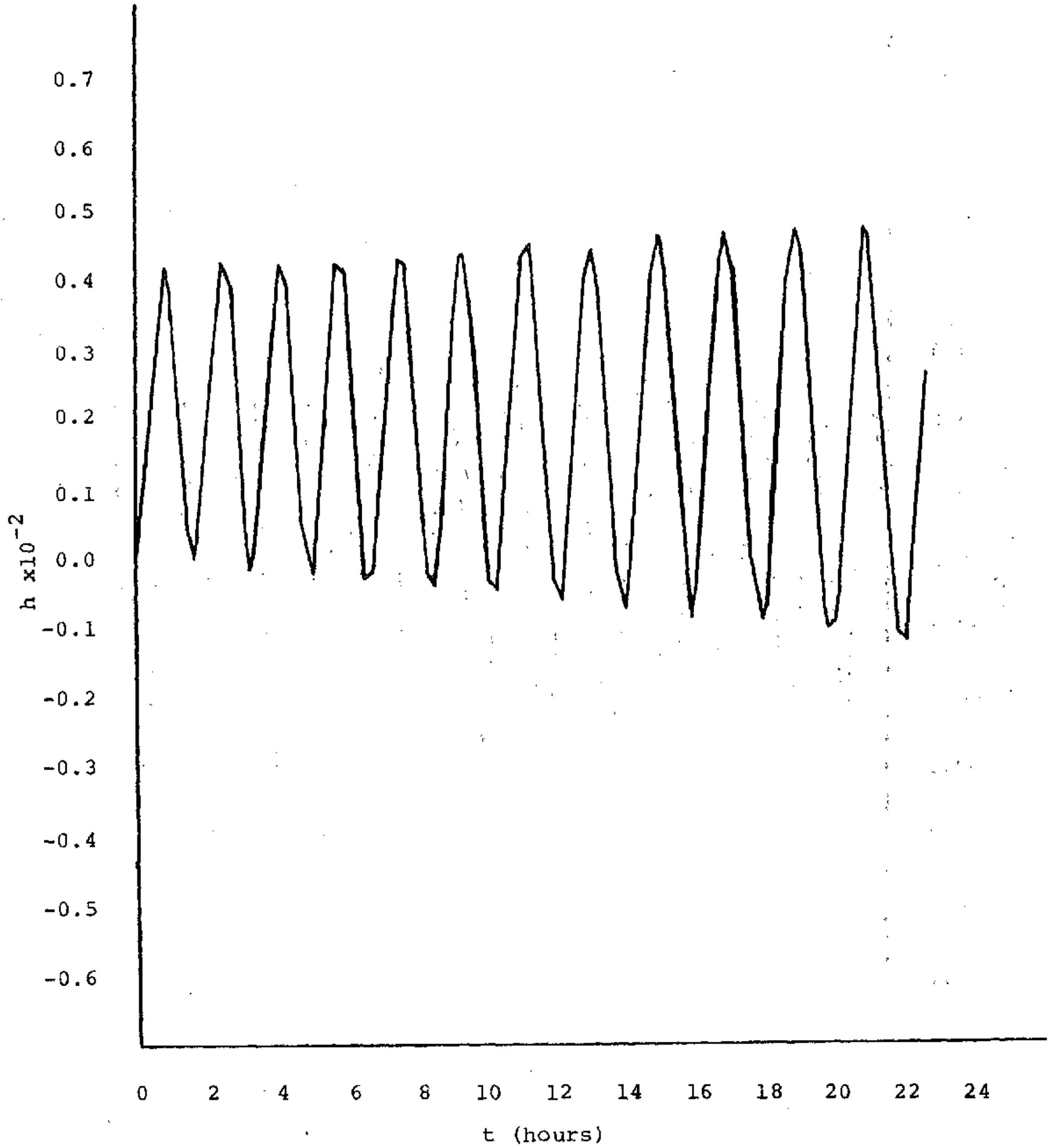
5.A Simulation Results

The minimum time deterministic control problem which generates the minimum time nominal trajectory defined in Section 3.A can only be solved by numerical methods. For the simulation in this research an approximate minimum time nominal trajectory is used. For the details of this approximate minimum time trajectory, the reader is referred to Shepperd [7]. Flying along this nominal trajectory the SEPS Spacecraft would reach the desired geosynchronous orbit from a near earth orbit. If the near earth orbit has a radius of 4300 miles and an inclination of 28 degrees, and if the desired geosynchronous orbit has a radius of 2600 miles and an inclination of 0 degrees, the nominal final time of this mission would be approximately 150 hours. In the results presented here only the first 22.64 hours of the mission are simulated. This nominal trajectory is pictured in Figures 5.1 to 5.6. The semi-major axis is increasing approximately linearly with time. This shows that the averaged radius of the orbit is in-



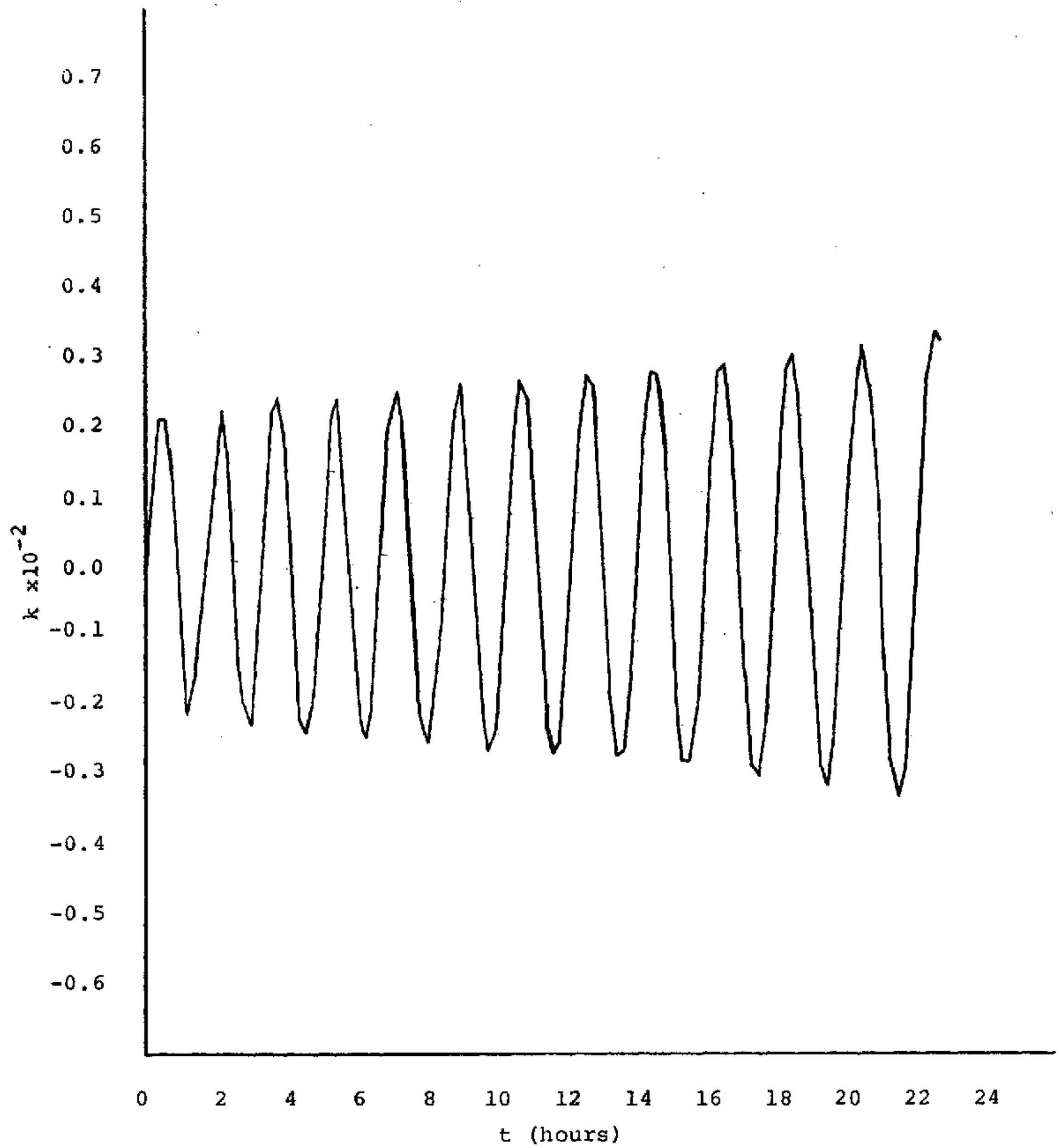
$(a)_{nominal}$ vs t

Figure 5.1



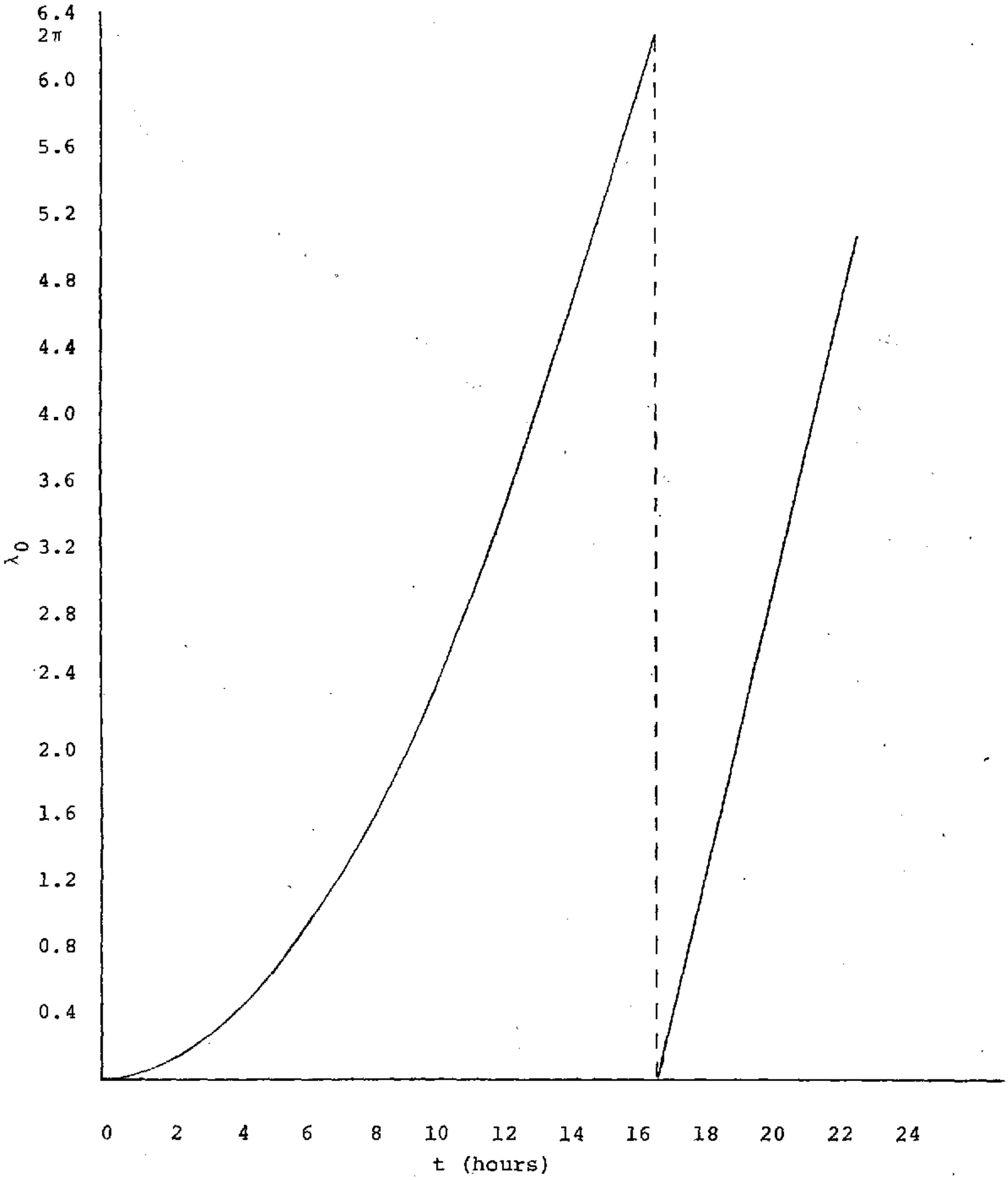
$(h)_{nominal}$ vs t

Figure 5.2



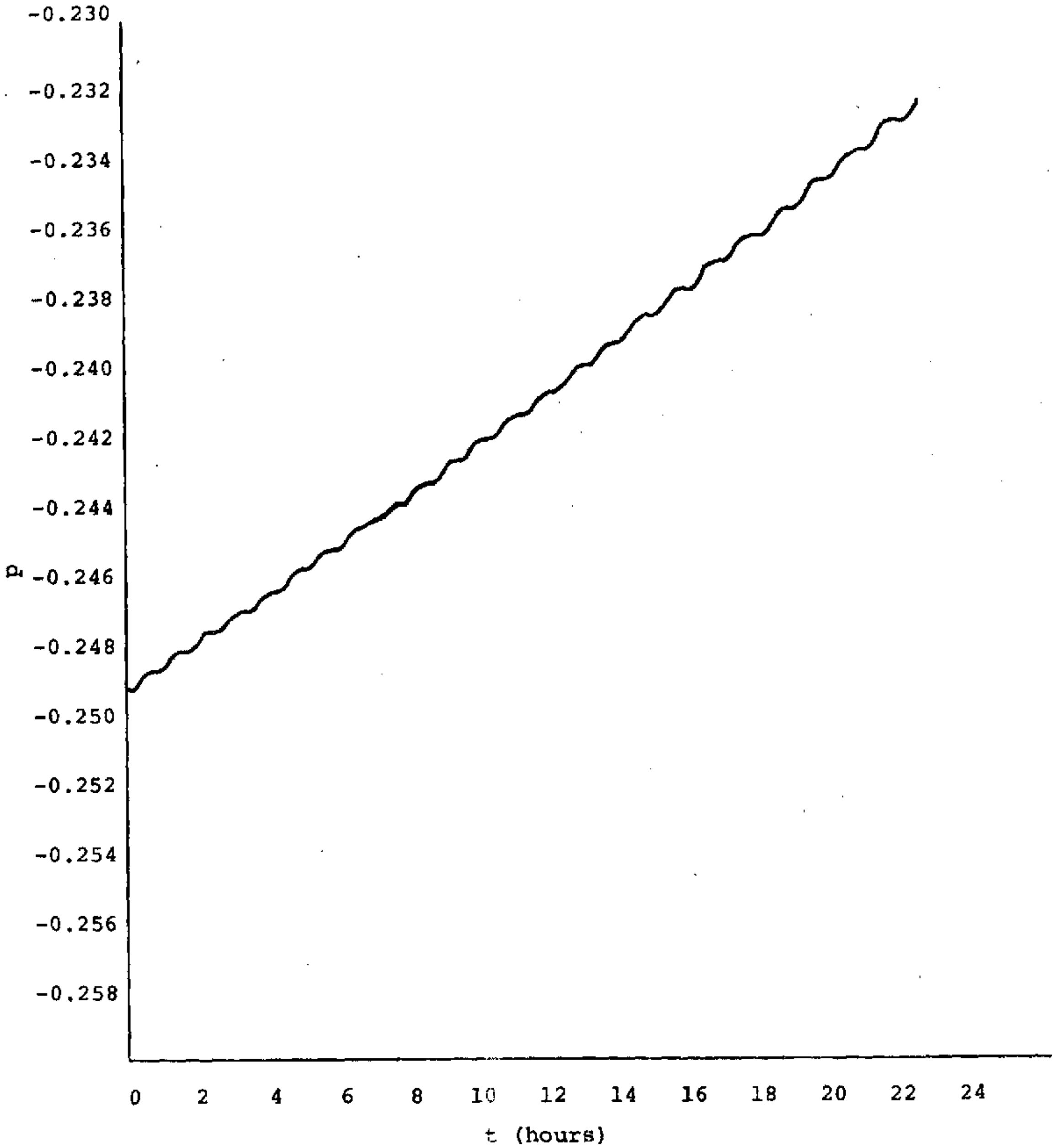
$(k)_{nominal}$ vs t

Figure 5.3



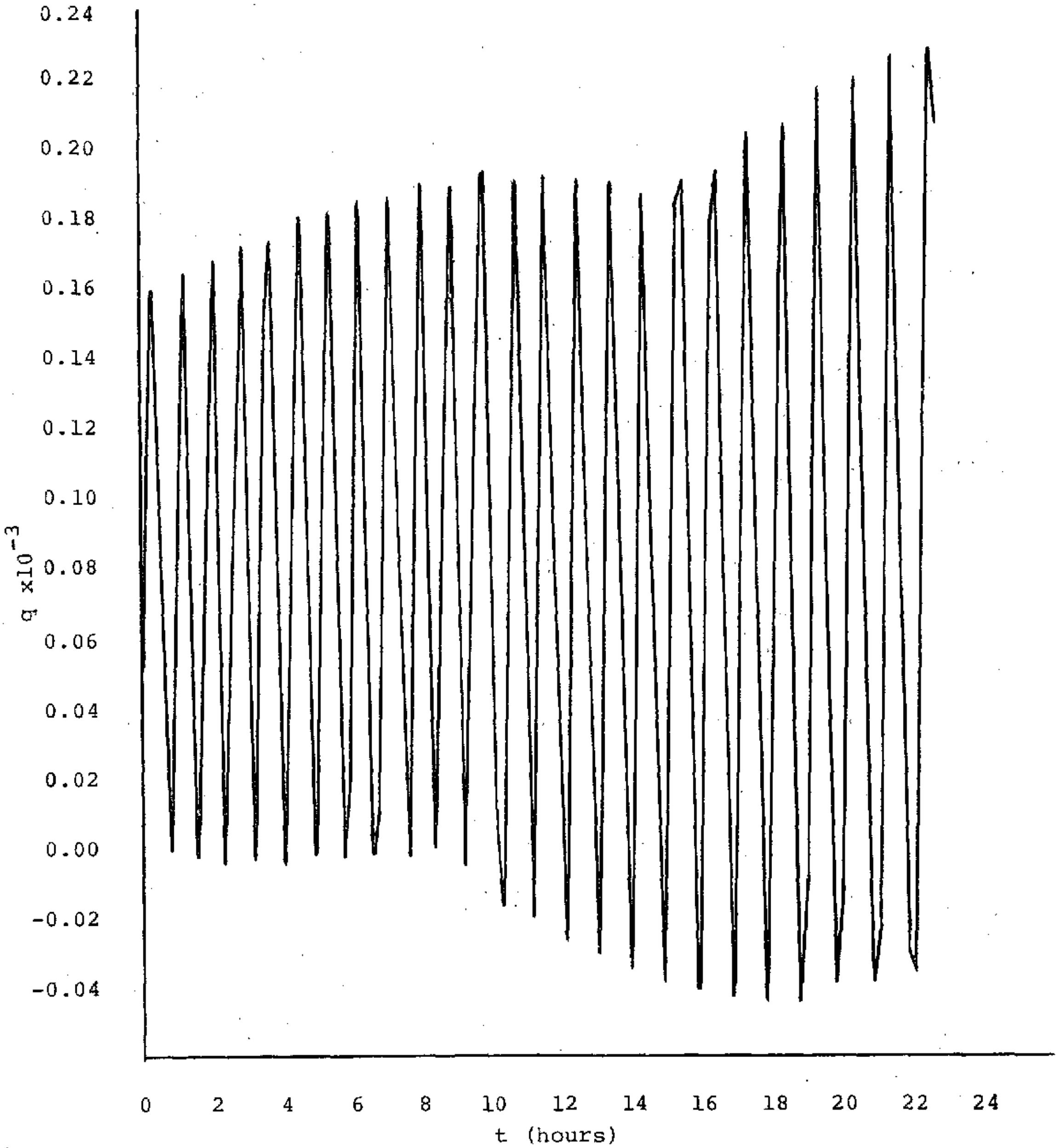
(λ_0) nominal vs t

Figure 5.4



$(p)_{nominal}$ vs t

Figure 5.5



$(q)_{nominal}$ vs t

Figure 5.6

creasing. The equinoctial elements h and k vary sinusuidally with an increasing amplitude. This increasing of amplitude showed that the averaged eccentricity of the orbit is increasing. The equinoctial element λ_0 is increasing monotonically from 0 radians to $(2\pi + 5.073)$ radians. This variation showed how the angular position of the SEPS Spacecraft in the orbit is changed by the engine thrust acceleration. Finally the variations of the equinoctial elements p and q showed that the inclination of the orbit is decreasing monotonically.

The values of the input variables used in this simulation are summarized as follows. The statistics of the initial state are

$$\bar{\underline{x}}(0) = \begin{bmatrix} 0.1085 \times 10' er \\ 0.0 \\ 0.0 \\ 0.0 \\ -0.249 \\ 0.0 \end{bmatrix} \tag{5.A.1}$$

$$m(0) = \begin{bmatrix} 0.196\text{x}10^{-5}er^2 & 0.0 & 0.169\text{x}10^{-5}er & 0.0 & er\ -0.153\text{x}10^{-22}er & 0.0 \\ 0.0 & er\ 0.139\text{x}10^{-5} & 0.0 & 0.143\text{x}10^{-5} & 0.0 & -0.161\text{x}10^{-22} \\ 0.169\text{x}10^{-5}er & 0.0 & 0.150\text{x}10^{-5} & 0.0 & -0.414\text{x}10^{-23} & 0.0 \\ 0.0 & er\ 0.143\text{x}10^{-5} & 0.0 & 0.154\text{x}10^{-5} & 0.0 & 0.480\text{x}10^{-7} \\ -0.153\text{x}10^{-22}er & 0.0 & -0.414\text{x}10^{-23} & 0.0 & 0.6102\text{x}10^{-7} & 0.0 \\ 0.0 & er\ -0.161\text{x}10^{-22} & 0.0 & 0.480\text{x}10^{-7} & 0.0 & 0.103\text{x}10^{-6} \end{bmatrix} \tag{5.A.2}$$

where er is earth-radii

$$er = 0.20925696 \times 10^{8} \text{ feet} \qquad (5.A.3)$$

The vector $\bar{\underline{x}}(0)$ is equivalent to a circular orbit with a radius of 4300 miles and an inclination of 28 degrees. The matrix m(0) is equivalent to the following standard deviations

$$(\sigma_r)_1 = 1 \text{ mile} \qquad (5.A.4)$$

$$(\sigma_r)_2 = 5 \text{ miles} \qquad (5.A.5)$$

$$(\sigma_r)_3 = 1 \text{ mile} \qquad (5.A.6)$$

$$(\sigma_v)_1 = 5 \text{ feet/second} \qquad (5.A.7)$$

$$(\sigma_v)_2 = 15 \text{ feet/second} \qquad (5.A.8)$$

$$(\sigma_v)_3 = 15 \text{ feet/second} \qquad (5.A.9)$$

where $(\sigma_r)_1$, $(\sigma_r)_2$, $(\sigma_r)_3$, $(\sigma_v)_1$, $(\sigma_v)_2$, $(\sigma_v)_3$ are the standard deviations of position and velocity in altitude, down range and cross track directions respectively. These statistics are typical of a spacecraft launch trajectory. The nominal final time is

$$t_f = 22.64 \text{ hours} \qquad (5.A.10)$$

The parameters defining the target set are

$$\underline{x}_f = \begin{bmatrix} 0.1329\times10^{1}\,er \\ 0.2670\times10^{-2} \\ 0.3221\times10^{-2} \\ 0.5073\times10^{1} \\ -0.2325 \\ 0.2068\times10^{-3} \end{bmatrix} \qquad (5.A.11)$$

$$(\delta x_f)_1 = 0.775 \text{ x}10^{-4}\text{er} \tag{5.A.12}$$

$$(\delta x_f)_2 = 0.858 \text{ x}10^{-4} \tag{5.A.13}$$

$$(\delta x_f)_3 = 0.971 \text{ x}10^{-4} \tag{5.A.14}$$

$$(\delta x_f)_5 = 0.594 \text{ x}10^{-4} \tag{5.A.15}$$

$$(\delta x_f)_6 = 0.106 \text{ x}10^{-f} \tag{5.A.16}$$

The values of the parameters δx_f defined the size of the target set. Since it is expected that the deviation between the true and nominal state at the nominal final time will not be less than the expected estimation error, the values δx_f in (5.A.12) to (5.A.16) are taken from the standard deviations of the corresponding diagonal elements of the estimation error covariance matrix $P(t_f)$. Note that the covariance matrix $P(t_f)$ is computed along the minimum time nominal trajectory which is not the same as the covariance matrix $P_i(t_f)$ computed using the equations of the extended Kalman filter. The thrust acceleration function is

$$u_m(t) = \frac{\varepsilon g_0}{(1-\frac{\varepsilon}{I_s}t)} \tag{5.A.17}$$

where g_0 is the surface gravity acceleration, I_s is the engine specific impulse and ε is the engine's initial thrust acceleration in terms of the g_0's

$$\varepsilon = 0.1 \text{ x}10^{-2} \tag{5.A.18}$$

$$I_s = 0.4 \text{ x}10^{4} \text{ sec} \tag{5.A.19}$$

$$g_0 = 32.0 \text{ feet/second}^2 \tag{5.A.20}$$

The parameters which represent the strength of the process noise are

$$N_1 = 0.1 \text{ x}10^{-3} \tag{5.A.21}$$

$$N_2 = 0.42 \text{ x}10^{-7} \tag{5.A.22}$$

The value of the parometer N_1 in (5.A.21) is equal to the square of 1 percent and the value of the parameter N_3 in (5.A.22) is equal to 1/2 of the square of 1/60 degree. A set of measurements is taken every half orbital period. Each set of measurements consist of one earth-diameter and two star-elevation measurements. The parameter which represents the strength of the measurement noise is

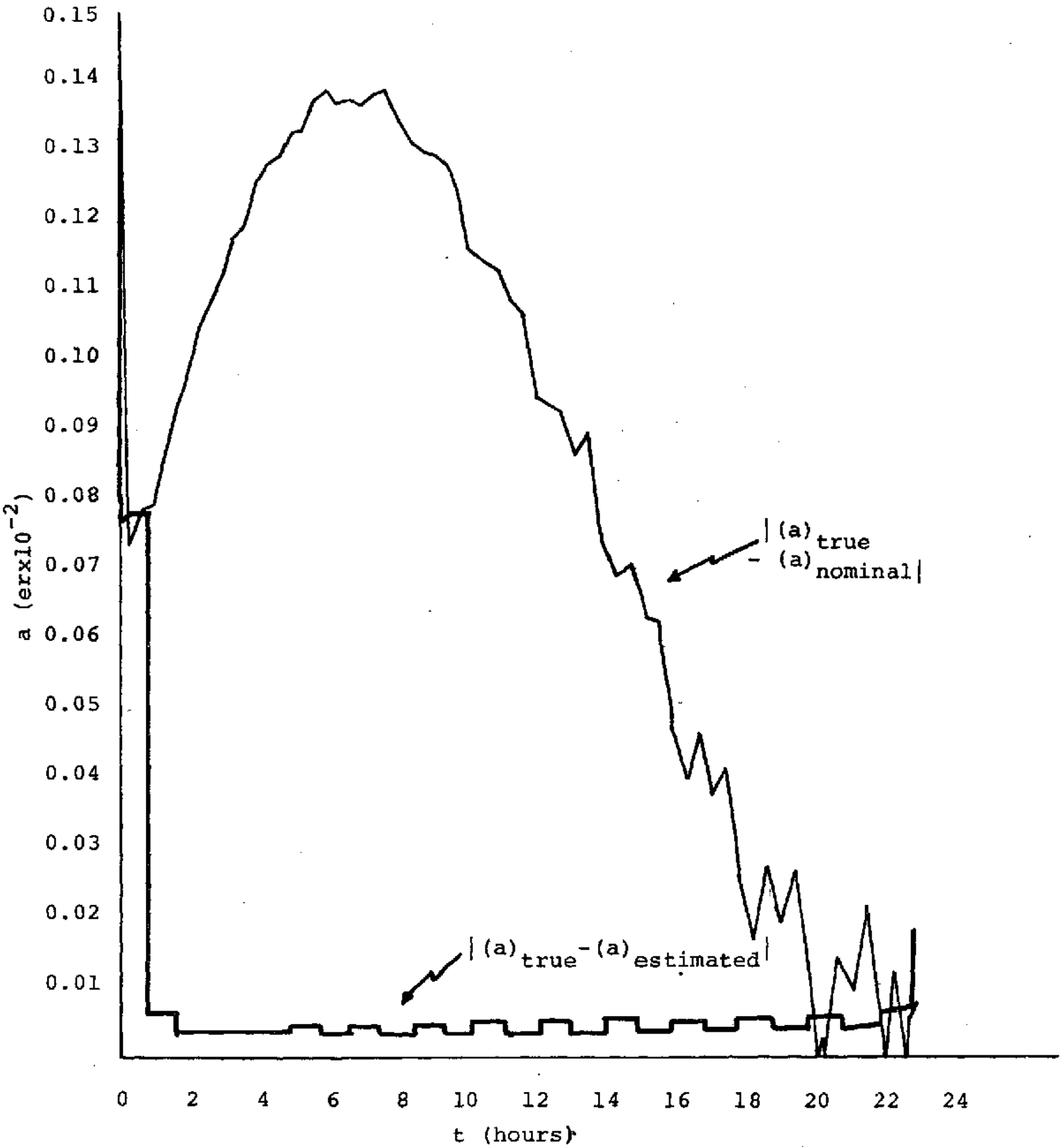
$$V_i = \begin{bmatrix} 0.84\text{x}10^{-7} & 0 & 0 \\ 0 & 0.84\text{x}10^{-7} & 0 \\ 0 & 0 & 0.84\text{x}10^{-7} \end{bmatrix} \qquad (5.A.23)$$

Note that $(0.84\text{x}10^{-7})$ is equal to the square of 1/60 degrees.

The results of this simulation are pictured in Figures 5.7 to 5.12 where the difference between the true state and nominal state, the difference between the true state and the estimated state as shown. At the nominal time t_f, the results are

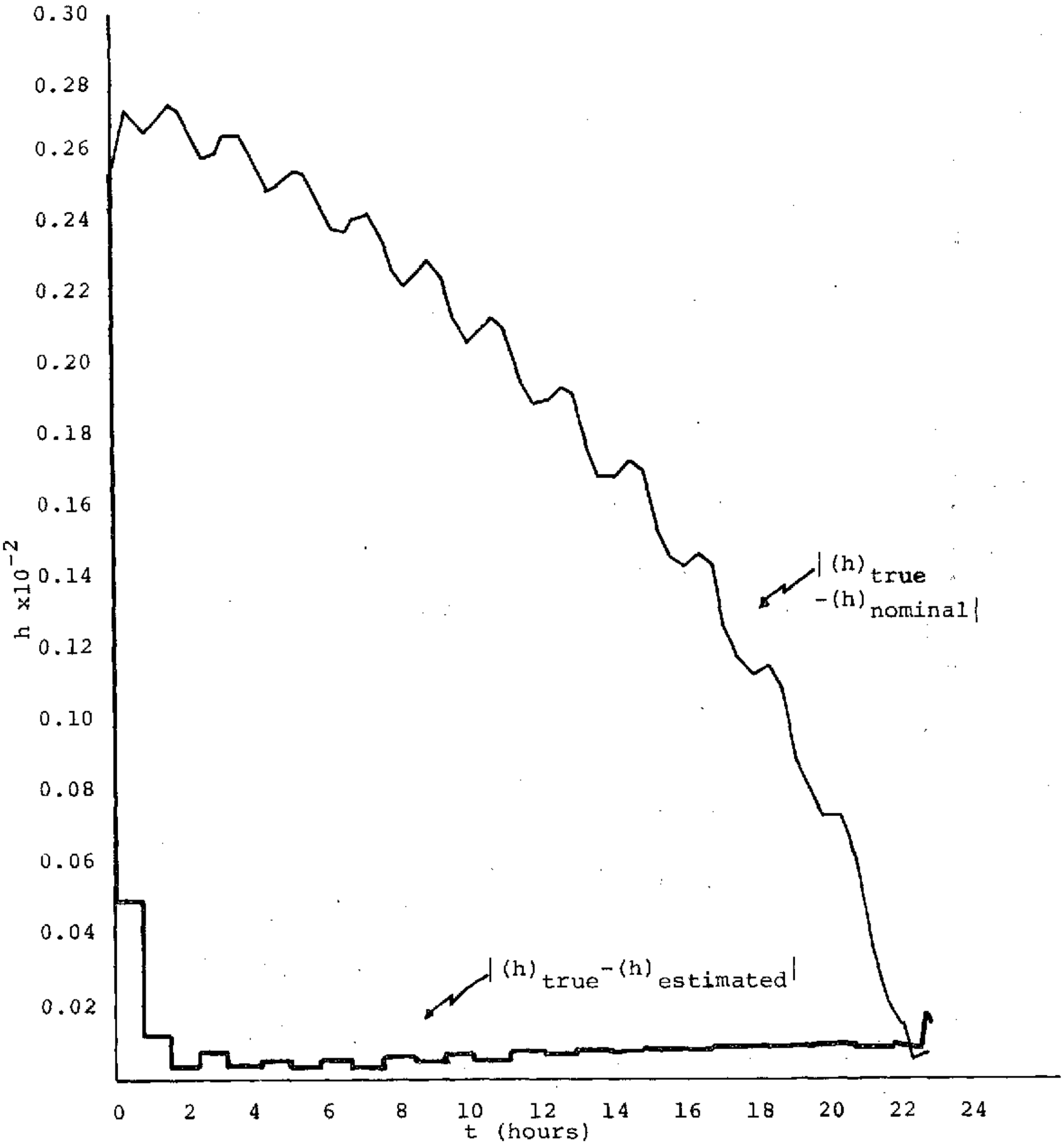
$$\underline{x}(t_f)_{\text{true}} - x(t_f)_{\text{nominal}} = \begin{bmatrix} 0.9232 \times 10^{-4}\text{er} \\ 0.7592 \times 10^{-4} \\ -0.4034 \times 10^{-4} \\ -0.6880 \times 10^{-1} \\ -0.2113 \times 10^{-4} \\ 0.6884 \times 10^{-4} \end{bmatrix} \qquad (5.A.24)$$

$$\underline{x}(t_f)_{\text{true}} - \underline{x}(t_f)_{\text{estimated}} = \begin{bmatrix} 0.1796 \times 10^{-3}\text{er} \\ 0.1423 \times 10^{-3} \\ 0.6247 \times 10^{-4} \\ 0.1330 \times 10^{-1} \\ 0.3895 \times 10^{-4} \\ 0.1145 \times 10^{-4} \end{bmatrix} \qquad (5.A.25)$$



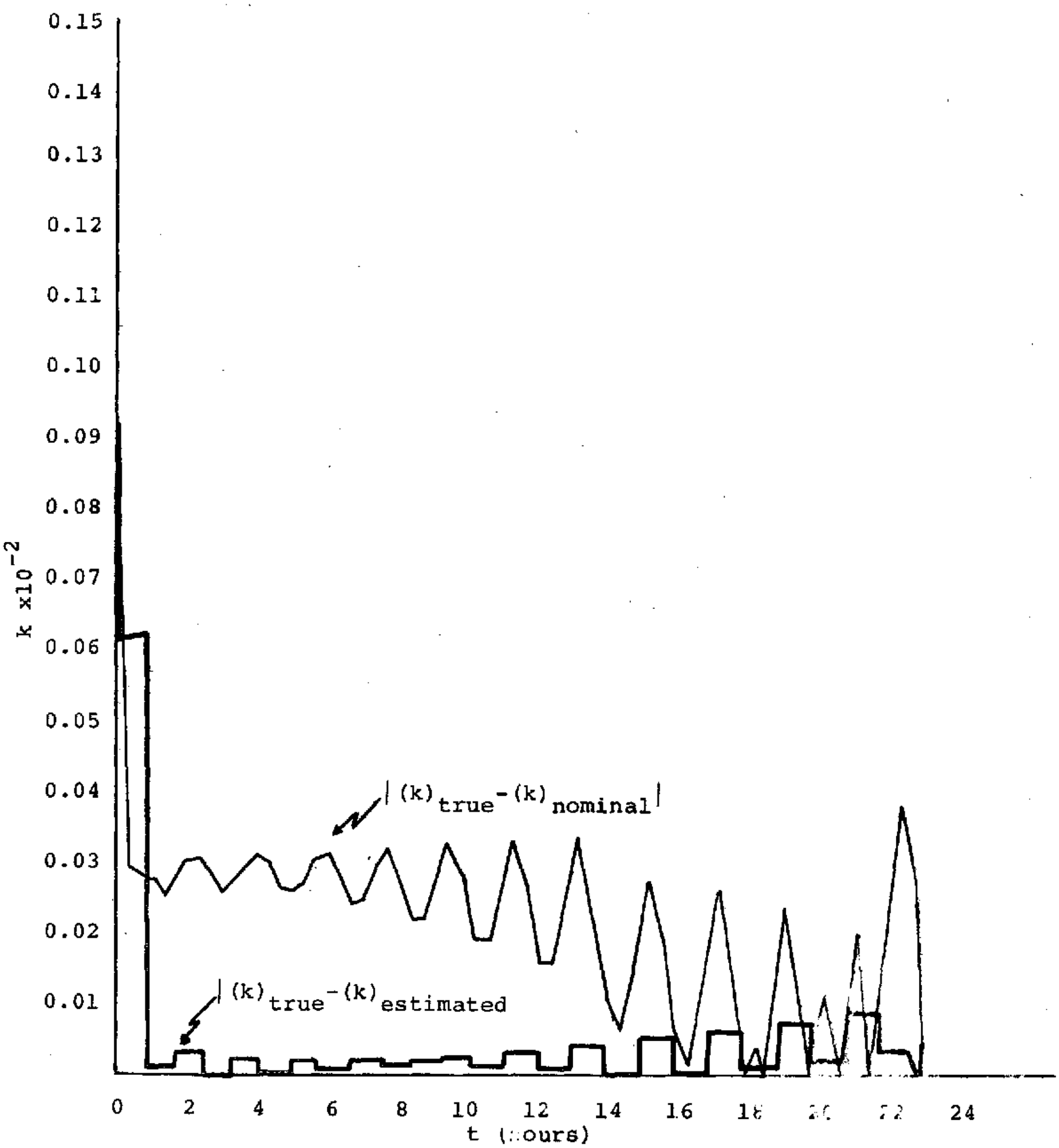
$|(a)_{true}-(a)_{nominal}|$ and $|(a)_{true}-(a)_{estimated}|$ vs t

Figure 5.7



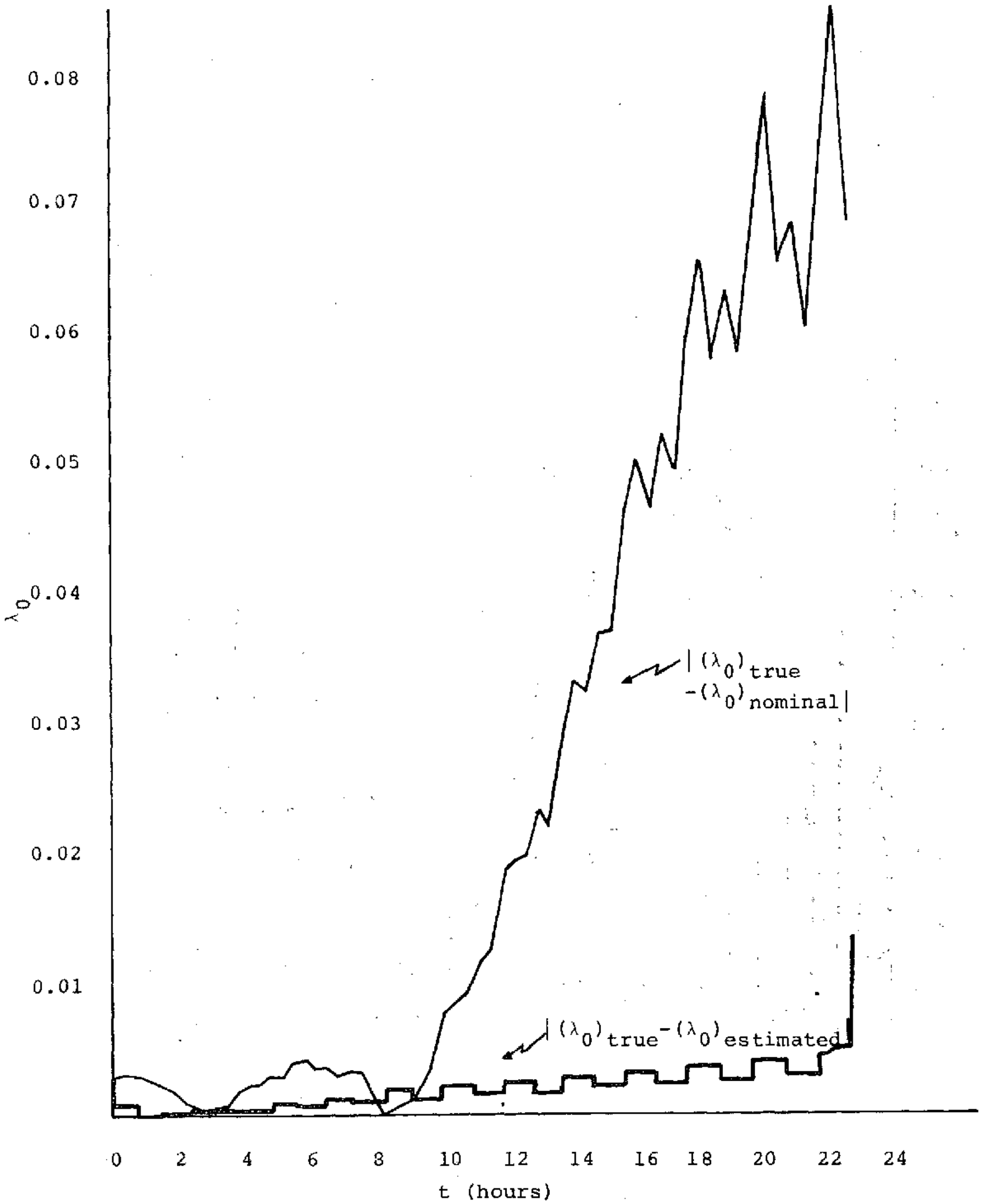
$|(h)_{true}-(h)_{nominal}|$ and $|(h)_{true}-(h)_{estimated}|$ vs t

Figure 5.8



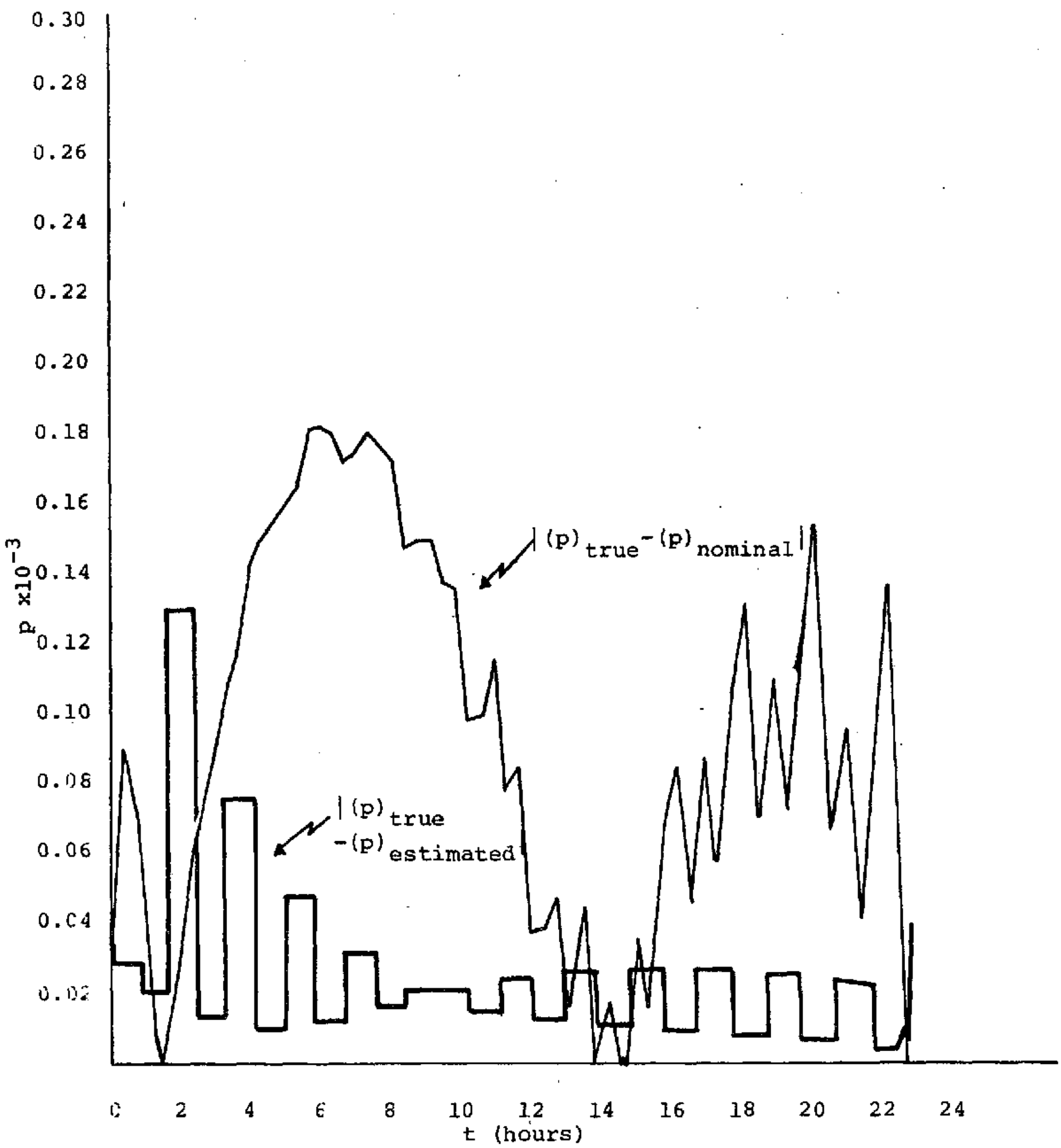
$|(k)_{true}-(k)_{nominal}|$ and $|(k)_{true}-(k)_{estimated}|$

Figure 5.9



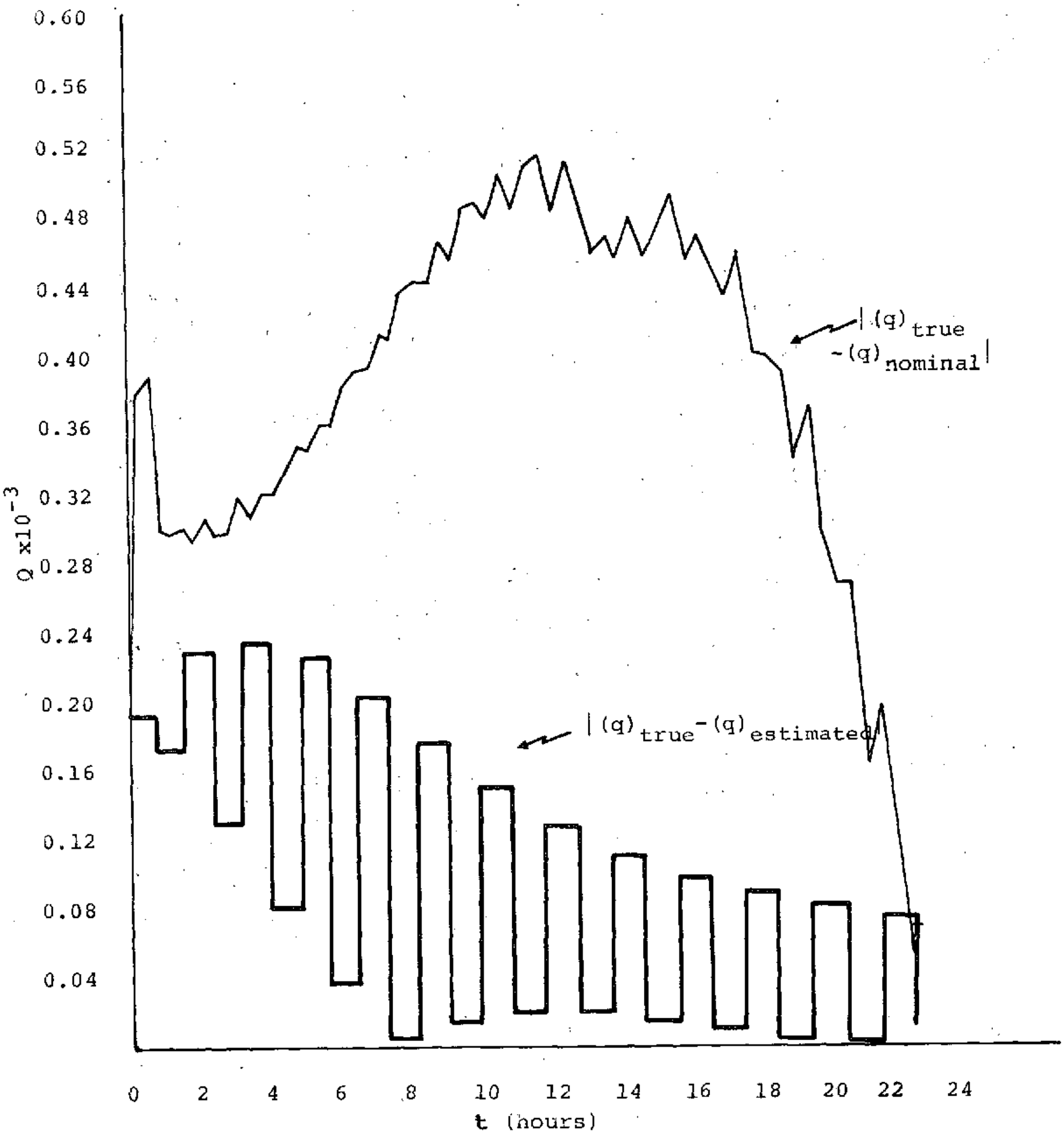
$|(\lambda_0)_{true} - (\lambda_0)_{nominal}|$ and $|(\lambda_0)_{true} - (\lambda_0)_{estimated}|$ vs k

Figure 5.10



$|(p)_{true}-(p)_{nominal}|$ and $|(p)_{true}-(p)_{estimated}|$ vs t

Figure 5.11



$|(q)_{true}-(q)_{nominal}|$ and $|(q)_{true}-(q)_{estimated}|$ vs k

Figure 5.12

The corresponding results in terms of the classical orbit elements are

$$a(t_f)_{true} - a(t_f)_{nominal} = 0.9232\times10^{-4}\,er \quad (5.A.26)$$

$$e(t_f)_{true} - e(t_f)_{nominal} = 0.1824\times10^{-4} \quad (5.A.27)$$

$$i(t_f)_{true} - i(t_f)_{nominal} = 0.4009\times10^{-4} \quad (5.A.28)$$

$$M_0(t_f)_{true} - M_0(t_f)_{nominal} = 0.8887\times10^{-1} \quad (5.A.29)$$

$$\omega(t_f)_{true} - \omega(t_f)_{nominal} = 0.1974\times10^{-1} \quad (5.A.30)$$

$$\Omega(t_f)_{true} - \Omega(t_f)_{nominal} = 0.2959\times10^{-3} \quad (5.A.31)$$

$$a(t_f)_{true} - a(t_f)_{estimated} = 0.1796\times10^{-3}\,er \quad (5.A.32)$$

$$e(t_f)_{true} - e(t_f)_{estimated} = 0.1397\times10^{-3} \quad (5.A.33)$$

$$i(t_f)_{true} - i(t_f)_{estimated} = -0.7390\times10^{-4} \quad (5.A.34)$$

$$M_0(t_f)_{true} - M_0(t_f)_{estimated} = 0.3179\times10^{-2} \quad (5.A.35)$$

$$\omega(t_f)_{true} - \omega(t_f)_{estimated} = 0.1651\times10^{-1} \quad (5.A.36)$$

$$\Omega(t_f)_{true} - \Omega(t_f)_{estimated} = -0.4904\times10^{-4} \quad (5.A.37)$$

5.B Discussion

The results showed that the SEPS Spacecraft was flying in a neighboring trajectory and reached a point close to the desired target set at the nominal final time t_f. Due to the presence of the uncertainties,the deviation between the true and nominal trajectory is not small throughout the flight except at time t_f. At time t_f, the deviation between the true and the nominal state is small. A comparison of (5.A.12) to (5.A.16) and (5.A.24) showed that the closed-loop system is performing reasonably well for this short mission. The values of $|h(t_f)_{true} - h(t_f)_{nominal}|$, $|k(t_f)_{true} - k(t_f)_{nominal}|$ and $|p(t_f)_{true} - p(t_f)_{nominal}|$ are less than the values of $(\delta x_f)_2$, $(\delta x_f)_3$ and $(\delta x_f)_5$ respectively. The value of $|a(t_f)_{true} - a(t_f)_{nominal}|$ is approximately 1.19 times the value of $(\delta x_f)_1$.

The value of $|q(t_f)_{true} - q(t_f)_{nominal}|$ is approxiamtely 64.94 times the value of $(\delta x_f)_6$. This large ratio is due to the fact that $(\delta x_f)_6$ is very small and the actual estimation error $|q(t_f)_{true} - q(t_f)_{estimated}|$ is approxiamately 10.6 times the value of $(\delta x_f)_6$. However, since the equinoctial elements p and q should have the same character, a comparison of the values of $|p(t_f)_{true} - p(t_f)_{nominal}|$ and $|q(t_f)_{true} - q(t_f)_{nominal}|$ in (5.A.24) showed that the closed-loop system is still performing reasonably well. The value of $|(\lambda_0)_{true} - (\lambda_0)_{nominal}|$ is not small for this midcourse flight since this equinoctial element is not included in the exponential cost criterion.

The deviation between the true and estimated state is also presented in Figures 5.7 to 5.12. Between two measurements the estimation errors are approximately constant. At a measurement the estimation errors have discontinuities. These estimation errors tend to increase slowly with time. This indicated that for a longer mission some more accurate measurements such as ground based tracking must be used to reduce these estimation errors. At these high accuracy ground based measurements,the minimum time nominal trajectory, the feedback control gain matrix $\Lambda(t_j)$ and the vector $\underline{b}(t_j)$ could also be updated. If these ground based updates are included in the miscourse guidance and navigation of the SEPS Spacecraft, the closed-loop system developed in this research should also perform well for a longer mission.

The most important advantage of using the LEG guidance law in the closed-loop system is that the weighting matrices L_j and Q_f can be chosen to achieve desired system performance. This fact is indicated by the simulation results. In the linear-quadratic-gaussian (LQG) problem [1], these weighting matrices usually must be obtained by iterations to achieve the desired system performance.

CHAPTER VI

CONCLUSIONS

A practical and efficient midcourse guidance and navigation sys system for the SEPS Spacecraft has been developed in this research. To reach the target set in minimum time the SEPS Spacecraft always utilizes full thrust magnitude. The thrusting direction of the engine is determined by the solution of the LEG terminal cost problem. The LEG approach provides a systematic way of determining weighting matrices for problems involving bounds and the control system design did not require many iterations, as is typically the case when the LQG approach is used. The solution of this problem, which is the guidance law, determines the control as a linear function of the current state estimate. Using this guidance law, the SEPS Spacecraft will fly in trajectory neighbouring the minimum time trajectory. To take into account this fact, the extended Kalman filter is used for the navigation of the SEPS Spacecraft.

The simulation results of a short mission have indicated that this closed-loop system is very efficient in bring the SEPS Spacecraft to the target set. However, these results have also indicated that the state estimation errors tend to increase slowly with time. For a long mission this means that the navigation system would have very poor state estimation and consequently the closed-loop system would have very poor performance. Therefore it is concluded that if this system is to be used effectively for a long mission, some more accurate ground based measurements and nominal trajectory updates must be included in the guidance and navigation of the SEPS Spacecraft.

The determination of an efficient measurement schedule, the frequency of high accuracy ground based measurement and nominal trajectory updates can be carried for further study to improve the effectiveness and performance of this closed-loop system. The possibilities of using this closed-loop system for terminal guidance and navigation of the SEPS can also be investigated.

APPENDIX A

THE MATRIX $\frac{\partial \underline{x}}{\partial \underline{r}}$

The matrix $\frac{\partial \underline{x}}{\partial \underline{r}}$ is given by

$$\frac{\partial \underline{x}}{\partial \underline{r}} = [\frac{\partial a^T}{\partial \underline{r}} \; \frac{\partial h^T}{\partial \underline{r}} \; \frac{\partial k^T}{\partial \underline{r}} \; \frac{\partial \lambda_0{}^T}{\partial \underline{r}} \; \frac{\partial p^T}{\partial \underline{r}} \; \frac{\partial q^T}{\partial \underline{r}}] \tag{A-1}$$

where

$$\frac{\partial a^T}{\partial \underline{r}} = \frac{a^2}{r^3}\, \underline{r} \tag{A-2}$$

$$\frac{\partial h^T}{\partial \underline{r}} = - \frac{\sqrt{1-h^2-k^2}}{ma^2} [(\frac{\partial \dot{X}_1}{\partial k} + h\beta m \frac{a^3}{r^3} X_1)\underline{f} + (\frac{\partial \dot{Y}_1}{\partial k} + h\beta m \frac{a^3}{r^3} Y_1)\underline{g}]$$
$$- \frac{k(q\dot{Y}_1 - p\dot{X}_1)}{ma^2\sqrt{1-h^2-k^2}}\, \underline{w} \tag{A-3}$$

$$\frac{\partial k^T}{\partial \underline{r}} = \frac{\sqrt{1-h^2-k^2}}{ma^2} [(\frac{\partial \dot{X}_1}{\partial h} - k\beta m \frac{a^3}{r^3} X_1)\underline{f} + (\frac{\partial \dot{Y}_1}{\partial h} - k\beta m \frac{a^3}{r^3} Y_1)\underline{g}]$$
$$+ \frac{h(q\dot{Y}_1 - p\dot{X}_1)}{ma^2\sqrt{1-h^2-k^2}}\, \underline{w} \tag{A-4}$$

$$\frac{\partial \lambda_0{}^T}{\partial \underline{r}} = - \frac{1}{ma^2}(\underline{v} - \frac{3\mu t}{r^3}\, \underline{r}) - \frac{\sqrt{1-h^2-k^2}}{ma^2}\, \beta [(h \frac{\partial \dot{X}_1}{\partial h} + k \frac{\partial \dot{X}_1}{\partial k})\underline{f}$$
$$+ (h\frac{\partial \dot{Y}_1}{\partial h} + k \frac{\partial \dot{Y}_1}{\partial k})\underline{g}] - \frac{(q\dot{Y}_1 - p\dot{X}_1)}{ma^2\sqrt{1-h^2-k^2}}\, \underline{w} \tag{A-5}$$

$$\frac{\partial \underline{p}^T}{\partial \underline{r}} = - \frac{1+p^2+q^2}{ma^2\sqrt{1-h^2k^2}} \dot{Y}_1 \underline{w} \tag{A-6}$$

$$\frac{\partial \underline{q}^T}{\partial \underline{r}} = - \frac{1+p^2+q^2}{ma^2\sqrt{1-h^2-k^2}} \dot{X}_1 \underline{w} \tag{A-7}$$

$$\frac{\partial \dot{X}_1}{\partial h} = \left(\frac{\partial \underline{v}}{\partial h}\right)^T \underline{f} \tag{A-8}$$

$$\frac{\partial \dot{X}_1}{\partial k} = \left(\frac{\partial \underline{v}}{\partial k}\right)^T \underline{f} \tag{A-9}$$

$$\frac{\partial \dot{Y}_1}{\partial h} = \left(\frac{\partial \underline{v}}{\partial h}\right)^T \underline{g} \tag{A-10}$$

$$\frac{\partial \dot{Y}_1}{\partial k} = \left(\frac{\partial \underline{v}}{\partial k}\right)^T \underline{g} \tag{A-11}$$

The partial deriviatives $\frac{\partial \underline{v}}{\partial h}$, $\frac{\partial \underline{v}}{\partial k}$ are presented in Appendix C.

APPENDIX B

THE MATRIX $\frac{\partial \underline{r}}{\partial \underline{x}}$

The matrix $\frac{\partial \underline{r}}{\partial \underline{x}}$ is given by

$$\frac{\partial \underline{r}}{\partial \underline{x}} = [\ \frac{\partial \underline{r}}{\partial a}\ \ \frac{\partial \underline{r}}{\partial h}\ \frac{\partial \underline{r}}{\partial k}\ \ \frac{\partial \underline{r}}{\partial \lambda_0}\ \ \frac{\partial \underline{r}}{\partial p}\ \ \frac{\partial \underline{r}}{\partial q}\ ] \tag{B-1}$$

where

$$\frac{\partial \underline{r}}{\partial a} = \frac{1}{a}\ (\underline{r} - \frac{3}{2}\ t\ \underline{v}) \tag{B-2}$$

$$\frac{\partial \underline{r}}{\partial h} = \frac{\partial X_1}{\partial h}\ \underline{f} + \frac{\partial Y_1}{\partial h}\ \underline{g} \tag{B-3}$$

$$\frac{\partial \underline{r}}{\partial k} = \frac{\partial X_1}{\partial k}\ \underline{f} + \frac{\partial Y_1}{k}\ \underline{g} \tag{B-4}$$

$$\frac{\partial \underline{r}}{\partial \lambda_0} = \frac{\underline{v}}{m} \tag{B.5}$$

$$\frac{\partial \underline{r}}{\partial p} - \frac{2}{1+p^2+q^2}\ [q(Y_1\underline{f} - X_1\underline{g})-X_1\underline{w}] \tag{B.6}$$

$$\frac{\partial \underline{r}}{\partial q} = \frac{2}{1+p^2+q^2}\ [p(X_1\underline{g} - Y_1\underline{f}) + Y_1\underline{w}\ ] \tag{B.7}$$

APPENDIX C

THE MATRIX $\frac{\partial \underline{v}}{\partial \underline{x}}$

The matrix $\frac{\partial \underline{v}}{\partial \underline{x}}$ is given by

$$\frac{\partial \underline{v}}{\partial \underline{x}} = [\frac{\partial \underline{v}}{\partial a} \quad \frac{\partial \underline{v}}{\partial h} \quad \frac{\partial \underline{v}}{\partial k} \; \frac{\partial \underline{v}}{\partial \lambda_0} \; \frac{\partial \underline{v}}{\partial p} \; \frac{\partial \underline{v}}{\partial q}] \tag{C-1}$$

where

$$\frac{\partial \underline{v}}{\partial a} = - \frac{1}{2a} (\underline{v} - \frac{3\mu t}{r^3} \underline{r}) \tag{C-2}$$

$$\frac{\partial \underline{v}}{\partial h} = \frac{\partial \dot{X}_1}{\partial h} \underline{f} + \frac{\partial \dot{Y}_1}{\partial h} \underline{g} \tag{C-3}$$

$$\frac{\partial \underline{v}}{\partial k} = \frac{\partial \dot{X}_1}{\partial k} \underline{f} + \frac{\partial \dot{Y}_1}{\partial k} \underline{g} \tag{C-4}$$

$$\frac{\partial \underline{v}}{\partial \lambda 0} = -m \frac{a^3}{r^3} r \tag{C-5}$$

$$\frac{\partial \underline{v}}{\partial p} = \frac{2}{1+p^2+q^2} [q(\dot{Y}_1\underline{f} - \dot{X}_1\underline{g}) - \dot{X}_1 \underline{w}] \tag{C-6}$$

$$\frac{\partial \underline{v}}{\partial q} = \frac{2}{1+p^2q^2} [p(\dot{X}_1\underline{g} - \dot{Y}_1\underline{f}) + \dot{Y}_1\underline{w}] \tag{C-7}$$

$$\frac{\partial \dot{X}_1}{\partial h} = \frac{ma^2}{r} [(\frac{h^2\beta^3}{1-\beta} + \beta)(1-\frac{r}{a}) - \frac{ah\beta}{r} \cos F(\lambda-F) + h\beta \sin F + \frac{a}{r} \cos^2 F]$$

$$- \frac{\dot{X}_1 a}{r} [\frac{a}{r} \cos F (\lambda-F) - \sin F] \tag{C-8}$$

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$$\frac{\partial \dot{X}_1}{\partial k} = \frac{ma^2}{r}[\frac{hk\beta^3}{1-\beta}\ (1-\frac{r}{a}) + \frac{ah\beta}{r} \sin F\ (\lambda-F) + \cos F\ (h\beta - \frac{a}{r} \sin F)]$$

$$+ \frac{a\dot{X}_1}{r}\ [\frac{a}{r} \sin F\ (\lambda-F) + \cos F] \qquad \text{(C-9)}$$

$$\frac{\partial \dot{Y}_1}{\partial h} = \frac{ma^2}{r}\ [\frac{hk\beta^3}{1-\beta}(\frac{r}{a} - 1) + \frac{ak\beta}{r} \cos F\ (\lambda-F) - \sin F\ (k\beta - \frac{a}{r} \cos F)]$$

$$- \frac{a\dot{Y}_1}{r}\ [\frac{a}{r} \cos F\ (\lambda-F) - \sin F] \qquad \text{(C-10)}$$

$$\frac{\partial \dot{Y}_1}{\partial k} = \frac{ma^2}{r}[(\frac{k^2\beta^3}{1-\beta} + \beta)\ (\frac{r}{a} - 1) - \frac{ak\beta}{r} \sin F(\lambda-F) - k\beta \cos F - \frac{a}{r} \sin^2 F]$$

$$+ \frac{a\dot{Y}_1}{r}[\frac{a}{r} \sin F(\lambda-F) + \cos F] \qquad \text{(C-11)}$$

$$\lambda = \lambda_0 + mt \qquad \text{(C-12)}$$

APPENDIX D

THE MATRIX $A(\underline{x}_0, \underline{u}_0, t)$

The matrix $A(\underline{x}_0, \underline{u}_0, t)$ is given by

$$A(x_0, u_0, t) = [\frac{\partial}{\partial \underline{x}} G(\underline{x},\underline{t})u]_{\underline{x}_0,\underline{u}_0}$$

$$\begin{bmatrix}
\frac{\partial}{\partial a}(\frac{\partial a}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial h}(\frac{\partial a}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial k}(\frac{\partial a}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial \lambda_0}(\frac{\partial a}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial p}(\frac{\partial a}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial q}(\frac{\partial a}{\partial \underline{v}}\underline{u}) \\
\frac{\partial}{\partial a}(\frac{\partial h}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial h}(\frac{\partial h}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial k}(\frac{\partial h}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial \lambda_0}(\frac{\partial h}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial p}(\frac{\partial h}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial q}(\frac{\partial h}{\partial \underline{v}}\underline{u}) \\
\frac{\partial}{\partial a}(\frac{\partial k}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial h}(\frac{\partial k}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial k}(\frac{\partial k}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial \lambda_0}(\frac{\partial k}{\partial v}\underline{u}) & \frac{\partial}{\partial p}(\frac{\partial k}{\partial v}\underline{u}) & \frac{\partial}{\partial q}(\frac{\partial k}{\partial v}\underline{u}) \\
\frac{\partial}{\partial a}(\frac{\partial \lambda_0}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial h}(\frac{\partial \lambda_0}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial k}(\frac{\partial \lambda_0}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial \lambda_0}(\frac{\partial \lambda_0}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial p}(\frac{\partial \lambda_0}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial q}(\frac{\partial \lambda_0}{\partial \underline{v}}\underline{u}) \\
\frac{\partial}{\partial a}(\frac{\partial p}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial h}(\frac{\partial p}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial k}(\frac{\partial p}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial \lambda_0}(\frac{\partial p}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial p}(\frac{\partial p}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial q}(\frac{\partial p}{\partial \underline{v}}\underline{u}) \\
\frac{\partial}{\partial a}(\frac{\partial q}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial h}(\frac{\partial q}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial k}(\frac{\partial q}{\partial \underline{v}} u) & \frac{\partial}{\partial \lambda_0}(\frac{\partial q}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial p}(\frac{\partial q}{\partial \underline{v}}\underline{u}) & \frac{\partial}{\partial q}(\frac{\partial q}{\partial \underline{v}}\underline{u})
\end{bmatrix}_{\underline{x}_0,\underline{u}_0} \qquad \text{(D-1)}$$

where

$$\frac{\partial}{\partial a}(\frac{\partial a}{\partial \underline{v}}\underline{u}) = (\frac{4a}{\mu}\underline{v} + \frac{2a^2}{\mu}\frac{\partial \underline{v}}{\partial a})^T \underline{u} \qquad \text{(D-2)}$$

$$\frac{\partial}{\partial h}\left(\frac{\partial a}{\partial \underline{v}}\,\underline{u}\right) = \frac{2}{m^2 a}\,\frac{\partial \underline{v}^T}{\partial h}\,\underline{u} \tag{D-3}$$

$$\frac{\partial}{\partial k}\left(\frac{\partial a}{\partial \underline{v}}\,\underline{u}\right) = \frac{2}{m^2 a}\,\frac{\partial \underline{v}^T}{\partial k}\,\underline{u} \tag{D-4}$$

$$\frac{\partial}{\partial \lambda_0}\left(\frac{\partial a}{\partial \underline{v}}\,\underline{u}\right) = \frac{2}{ma^2}\,\frac{\partial \underline{v}^T}{\partial \lambda_0}\,\underline{u} \tag{D-5}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial a}{\partial \underline{v}}\,\underline{u}\right) = \frac{2}{ma^2}\,\frac{\partial \underline{v}^T}{\partial p}\,\underline{u} \tag{D-6}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial a}{\partial \underline{v}}\,\underline{u}\right) = \frac{2}{ma^2}\,\frac{\partial \underline{v}^T}{\partial q}\,\underline{u} \tag{D-7}$$

$$\begin{aligned}\frac{\partial}{\partial a}\left(\frac{\partial h}{\partial \underline{v}}\,\underline{u}\right) = \Bigg\{ & -\frac{1}{2a}\,\frac{\partial h}{\partial \underline{v}}^T + \frac{\sqrt{1-h^2-k^2}}{ma^2}\left[\left(\frac{\partial^2 X_1}{\partial a \partial k} - \frac{h\beta}{m}\,\frac{\partial \dot{X}_1}{\partial a} - \frac{3}{2}\,\frac{h\beta}{ma}\,\dot{X}_1\right)\underline{f}\right. \\ & \left. + \left(\frac{\partial^2 Y_1}{\partial a \partial k} - \frac{h\beta}{m}\,\frac{\partial \dot{Y}_1}{\partial a} - \frac{3}{2}\,\frac{h\beta}{ma}\,\dot{Y}_1\right)\underline{g}\right] \\ & + \frac{k\left(q\,\frac{\partial Y_1}{\partial a} - p\,\frac{\partial X_1}{\partial a}\right)}{ma^2\sqrt{1-h^2k^2}}\,\underline{w}\Bigg\}^T \underline{u}\end{aligned} \tag{D-8}$$

$$\begin{aligned}\frac{\partial}{\partial}\left(\frac{\partial h}{\partial \underline{v}}\,\underline{u}\right) = \Bigg\{ & -\frac{h}{ma^2\sqrt{1-h^2k^2}}\left[\left(\frac{\partial X_1}{\partial k} - h\beta\,\frac{\dot{X}_1}{m}\right)\underline{f} + \left(\frac{\partial Y_1}{\partial k} - h\beta\,\frac{\dot{Y}_1}{m}\right)\underline{g}\right] \\ & + \frac{hk(qY_1 - pX_1)}{ma^2(1-h^2k^2)^{1.5}}\,\underline{w} + \frac{\sqrt{1-h^2k^2}}{ma^2}\left[\left(\frac{\partial^2 X_1}{\partial h \partial k} - \beta\,\frac{\dot{X}_1}{m}\right.\right. \\ & \left. - \frac{h^2\beta^3}{m(1-\beta)}\,\dot{X}_1 - \frac{h\beta}{m}\,\frac{\partial \dot{X}_1}{\partial h}\right)\underline{f} + \left(\frac{\partial^2 Y_1}{\partial h \partial k} - \beta\,\frac{\dot{Y}_1}{m}\right. \\ & \left.\left. - \frac{h^2\beta^3}{m(1-\beta)}\,\dot{Y}_1 - \frac{h\beta}{m}\,\frac{\partial \dot{Y}_1}{\partial h}\right)\underline{g}\right] + \frac{k\left(q\,\frac{\partial Y_1}{\partial h} - p\,\frac{\partial X_1}{\partial h}\right)}{ma^2\sqrt{1-h^2k^2}}\,\underline{w}\Bigg\}^T \underline{u}\end{aligned} \tag{D-9}$$

$$\frac{\partial}{\partial k}\left(\frac{\partial h}{\partial \underline{v}}\ \underline{u}\right) = \Bigg\{ - \frac{k}{ma^2\sqrt{1-h^2k^2}} \left[\left(\frac{\partial X_1}{\partial k} - h\beta\ \frac{\dot{X}_1}{m}\right)\underline{f} + \left(\frac{\partial Y_1}{\partial k} - h\beta\ \frac{\dot{Y}_1}{m}\right)\underline{g}\right]$$

$$+ \frac{(1-h^2)\ (qY_1-pX_1)}{ma^2\ \ (1-h^2k^2)^{1.5}}\ \underline{w} + \frac{\sqrt{1-h^2k^2}}{ma^2} \left[\left(\frac{\partial^2 X_1}{\partial k^2} - \frac{hk\beta^3}{m(1-\beta)}\dot{X}_1 - \frac{h\beta}{m}\ \frac{\partial \dot{X}_1}{\partial k}\right)\underline{f}\right.$$

$$\left. + \left(\frac{\partial^2 Y_1}{\partial k^2} - \frac{hk\beta^3}{m(1-\beta)}\ \dot{Y}_1 - \frac{h\beta}{m}\ \frac{\partial \dot{Y}_1}{\partial k}\right) g\right]$$

$$+ \frac{k\left(q\ \frac{\partial Y_1}{k} - p\ \frac{\partial X_1}{k}\right)}{ma^2\sqrt{1-h^2k^2}}\ \underline{w}\Bigg\}^T\ \underline{u} \qquad \text{(D-10)}$$

$$\frac{\partial}{\partial \lambda_0}\left(\frac{\partial h}{\partial \underline{v}}\ \underline{u}\right) = \Bigg\{ \frac{\sqrt{1-h^2-k^2}}{ma^2} \left[\left(\frac{\partial^2 X_1}{\partial \lambda_0 \partial k} - \frac{h\beta}{m}\ \frac{\partial \dot{X}_1}{\partial \lambda_0}\right)\underline{f}\ + \right.$$

$$\left. + \left(\frac{\partial^2 Y_1}{\partial \lambda_0 \partial k} - \frac{h\beta}{m}\ \frac{\partial \dot{Y}_1}{\partial \lambda_0}\ g\right]+ \frac{k\left(q\ \frac{\partial Y_1}{\partial \lambda_0} - p\frac{\partial X_1}{\partial \lambda_0}\right)}{ma^2\sqrt{1-h^2k^2}}\ \underline{w}\Bigg\}^T\ \underline{u} \qquad \text{(D-11)}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial h}{\partial \underline{v}}\ \underline{u}\right) = \Bigg\{\frac{\sqrt{1-h^2-k^2}}{ma^2}\left[\left(\frac{\partial X_1}{\partial k} - h\beta\ \frac{\dot{X}_1}{m}\right)\frac{\partial \underline{f}}{\partial p} + \left(\frac{\partial Y_1}{\partial k} - h\beta\ \frac{\dot{Y}_1}{m}\right)\ \frac{\partial \underline{g}}{\partial p}\right]$$

$$+ \frac{k(qY_1 - pX_1)}{ma^2\sqrt{1-h^2-k^2}}\ \frac{\partial \underline{w}}{\partial p} - \frac{kX_1}{ma^2\sqrt{1-h^2-k^2}}\ \underline{w}\Bigg\}^T\ \underline{u} \qquad \text{(D-12)}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial h}{\partial \underline{v}}\ \underline{u}\right) = \Bigg\{\frac{\sqrt{1-h^2-k^2}}{ma^2}\left[\left(\frac{\partial X_1}{k} - h\beta\ \frac{\dot{X}_1}{m}\right)\ \frac{\partial \underline{f}}{\partial q} + \left(\frac{\partial Y_1}{\partial k} - h\beta\ \frac{\dot{Y}_1}{m}\right)\ \frac{\partial \underline{g}}{\partial p}\right]$$

$$+ \frac{k(qY_1-pX_1)}{ma^2\sqrt{1-h^2k^2}}\ \frac{\partial \underline{w}}{\partial q} + \frac{kY_1}{ma^2\sqrt{1-h^2-k^2}}\ \underline{w}\Bigg\}^T\ \underline{u} \qquad \text{(D-13)}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial k}{\partial \underline{v}}\ \underline{u}\right) = -\frac{1}{2a}\frac{\partial k}{\partial \underline{v}}^T - \frac{\sqrt{1-h^2-k^2}}{ma^2}\left[\frac{\partial^2 X_1}{\partial a \partial h} + \frac{k\beta}{m}\frac{\partial \dot{X}_1}{\partial a}\right.$$

$$\left. + \frac{3}{2}\frac{k\beta}{ma}\dot{X}_1)\underline{f} + (\frac{\partial^2 Y_1}{\partial a \partial h} + \frac{k\beta}{m}\frac{\partial \dot{Y}_1}{\partial a} + \frac{3}{2}\frac{k\beta}{ma}\dot{Y}_1)\underline{g}\right]$$

$$\left. - \frac{h(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a})}{ma^2\sqrt{1-h^2-k^2}}\underline{w}\right\}^T \underline{u} \qquad \text{(D-14)}$$

$$\frac{\partial}{\partial h}\left(\frac{\partial k}{\partial \underline{v}}\ \underline{u}\right) = \left\{\frac{h}{ma^2\sqrt{1-h^2-k^2}}\left[(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{m})\underline{f} + (\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{m})\underline{g}\right]\right.$$

$$- \frac{(1-k^2)(qY_1-pX_1)}{ma^2(1-h^2-k^2)^{3/2}}\underline{w} - \frac{\sqrt{1-h^2-k^2}}{ma^2}\left[\frac{\partial^2 X_1}{\partial h^2} + \frac{hk\beta^3}{m(1-\beta)}\dot{X}_1\right.$$

$$\left. + \frac{k}{m}\frac{\partial \dot{X}_1}{\partial a})\ \underline{f} + (\frac{\partial^2 Y_1}{\partial h^2} + \frac{hk\beta^3}{m(1-\beta)}\dot{Y}_1 + \frac{k\beta}{m}\frac{\partial \dot{Y}_1}{\partial h})\ \underline{g}\right]$$

$$\left. - \frac{h(q\frac{\partial Y_1}{\partial h} - p\frac{\partial X_1}{\partial h})}{na^2\sqrt{1-h^2-k^2}}\ \underline{w}\right\}^T \underline{u} \qquad \text{(D-15)}$$

$$\frac{\partial}{\partial k}\left(\frac{\partial k}{\partial \underline{v}}\ \underline{u}\right) = \left\{\frac{k}{ma^2\sqrt{1-h^2-k^2}}\left[(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{m})\ \underline{f} + (\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{m})\underline{g}\right]\right.$$

$$- \frac{hk(qY_1-pX_1)}{ma^2(1-h^2-k^2)^{3/2}}\underline{w} - \frac{\sqrt{1-h^2-k^2}}{ma^2}\left[(\frac{\partial^2 X_1}{\partial k \partial h} + \beta\frac{\dot{X}_1}{m}\right.$$

$$+ \frac{k^2\beta^3}{m(1-\beta)}\dot{X}_1 + \frac{k\beta}{m}\frac{\partial \dot{X}_1}{\partial k})\underline{f} + (\frac{\partial^2 Y_1}{\partial k \partial h} + \beta\frac{\dot{Y}_1}{m} + \frac{k^2\beta^3}{m(1-\beta)}\dot{Y}_1$$

$$\left.\left. + \frac{k\beta}{m}\frac{\partial \dot{Y}_1}{\partial k})\underline{g}\right] - \frac{h(q\frac{\partial Y_1}{\partial k} - p\frac{\partial X_1}{\partial k})}{ma^2\sqrt{1-h^2-k^2}}\underline{w}\right\}^T \underline{u} \qquad \text{(D-16)}$$

$$\frac{\partial}{\partial\lambda_0}\left(\frac{\partial k}{\partial \underline{v}}\ \underline{u}\right) = \left\{ -\ \frac{\sqrt{1-h^2-k^2}}{ma^2}\left[\left(\frac{\partial^2 X_1}{\partial\lambda_0\partial h} + \frac{k\beta}{m}\ \frac{\partial \dot{X}_1}{\partial\lambda_0}\right)\underline{f} + \left(\frac{\partial^2 Y_1}{\partial\lambda_0\partial h} + \frac{k\beta}{m}\ \frac{\partial \dot{Y}_1}{\partial\lambda_0}\right)\underline{g}\right] - \frac{h\left(q\ \frac{\partial Y_1}{\partial\lambda_0} - p\ \frac{\partial X_1}{\partial\lambda_0}\right)}{ma^2\sqrt{1-h^2-k^2}}\ \underline{w} \right\}^T \underline{u} \qquad \text{(D-17)}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial k}{\partial \underline{v}}\ \underline{u}\right) = \left\{ -\ \frac{\sqrt{1-h^2-k^2}}{ma^2}\left[\left(\frac{\partial X_1}{\partial h} + \frac{k\beta}{m}\ \dot{X}_1\right)\ \frac{\partial \underline{f}}{\partial p} + \left(\frac{\partial Y_1}{\partial h} + \frac{k\beta}{m}\ \dot{Y}_1\right)\frac{\partial \underline{g}}{\partial p}\right] - \frac{h(qY_1-pX_1)}{ma^2\sqrt{1-h^2-k^2}}\ \frac{\partial \underline{w}}{\partial p} + \frac{hX_1}{ma^2\sqrt{1-h^2-k^2}}\ \underline{w} \right\}^T \underline{u} \qquad \text{(D-18)}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial k}{\partial \underline{v}}\ \underline{u}\right) = \left\{ -\ \frac{\sqrt{1-h^2-k^2}}{ma^2}\left[\left(\frac{\partial X_1}{\partial h} + \frac{k\beta}{m}\ \dot{X}_1\right)\ \frac{\partial \underline{f}}{\partial q} + \left(\frac{\partial Y_1}{\partial h} + \frac{k\beta}{m}\ \dot{Y}_1\right)\ \frac{\partial \underline{g}}{\partial q}\right] - \frac{h(q\ Y_1-pX_1)}{ma^2\sqrt{1-h^2-k^2}}\ \frac{\partial \underline{w}}{\partial q} - \frac{hY_1}{ma^2\sqrt{1-h^2-k^2}}\ \underline{w} \right\}^T \underline{u} \qquad \text{(D-19)}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial\lambda_0}{\partial \underline{v}}\ \underline{u}\right) = \left\{ -\ \frac{1}{2a}\ \frac{\partial\lambda_0{}^T}{\partial \underline{v}} - \frac{2}{ma^3}\left(1-\ \frac{9}{4}\ \frac{\mu t^2}{r^3}\right)\underline{r} + \frac{3}{2}\ \frac{t}{ma^4}\ \underline{v} + \frac{\sqrt{1-h^2-k^2}}{ma^2}\ \beta\left[\left(\frac{h\partial^2 X_1}{\partial a\partial h} + k\ \frac{\partial^2 X_1}{\partial a\partial k}\right)\underline{f} + \left(h\ \frac{\partial^2 Y_1}{\partial a\partial h} + k\ \frac{\partial^2 Y_1}{\partial a\partial h}\right)\ \underline{g}\right] + \frac{\left(q\ \frac{\partial Y_1}{\partial a} - p\ \frac{\partial X_1}{\partial a}\right)}{ma^2\sqrt{1-h^2-k^2}}\ \underline{w} \right\}^T \underline{u}$$

(D-20)

$$\frac{\partial}{\partial h}\left(\frac{\partial \lambda_0}{\partial \underline{v}}\ \underline{u}\right) = \left\{ -\frac{2}{ma^2}\left(\frac{\partial \underline{r}}{\partial h} - \frac{3}{2}\ t\ \frac{\partial \underline{v}}{\partial h}\right) - \frac{h\beta^3}{ma^2(1-\beta)}\left[\left(h\ \frac{\partial X_1}{\partial h} + k\ \frac{\partial X_1}{\partial k}\right)\underline{f}\right.\right.$$

$$\left. + \left(h\ \frac{\partial Y_1}{\partial h} + k\ \frac{\partial Y_1}{\partial k}\right)\underline{g}\right] + \frac{h(qY_1 - pX_1)}{ma^2(1-h^2-k^2)^{3/2}}\ \underline{w}$$

$$+ \frac{\sqrt{1-h^2-k^2}}{ma^2}\ \beta\left[\left(\frac{\partial X_1}{\partial h} + h\ \frac{\partial^2 X_1}{\partial h^2} + k\ \frac{\partial^2 X_1}{\partial h \partial k}\right)\ \underline{f}\right.$$

$$\left. + \left(\frac{\partial Y_1}{\partial h} + h\ \frac{\partial^2 Y_1}{\partial h^2} + k\ \frac{\partial^2 Y_1}{\partial h \partial k}\right)\underline{g}\right]$$

$$\left. + \frac{\left(q\ \frac{\partial Y_1}{\partial h} - p\ \frac{\partial X_1}{\partial h}\right)}{ma^2\sqrt{1-h^2-k^2}}\ \underline{w}\right\}^T\ \underline{u} \qquad \text{(D-21)}$$

$$\frac{\partial}{\partial k}\left(\frac{\partial \lambda_0}{\partial \underline{v}}\ \underline{u}\right) = \left\{ -\frac{2}{ma^2}\left(\frac{\partial \underline{r}}{\partial k} - \frac{3}{2} t\ \frac{\partial \underline{v}}{\partial k}\right) - \frac{k\beta^3}{ma^2(1-\beta)}\left[\left(h\ \frac{\partial X_1}{\partial h} + k\ \frac{\partial X_1}{\partial k}\right)\ \underline{f}\right.\right.$$

$$\left. + \left(h\ \frac{\partial Y_1}{\partial h} + k \frac{\partial Y_1}{\partial k}\right)\underline{g}\right] + \frac{k(qY_1 - pX_1)}{ma^2(1-h^2-k^2)^{1.5}}\ \underline{w}$$

$$+ \frac{\sqrt{1-h^2-k^2}}{ma^2}\beta\left[h\ \frac{\partial^2 X_1}{\partial k\, \partial h} + \frac{\partial X_1}{\partial k} + k\ \frac{\partial^2 X_1}{\partial k^2}\right)\underline{f}$$

$$\left. + \left(h\ \frac{\partial^2 Y_1}{\partial k\, \partial h} + \frac{\partial Y_1}{\partial k} + k\ \frac{\partial^2 Y_1}{\partial k^2}\right)\ \underline{g}\right]$$

$$\left. + \frac{\left(q\ \frac{\partial Y_1}{\partial k} - p\ \frac{\partial X_1}{\partial k}\right)}{ma^2\sqrt{1-h^2-k^2}}\ \underline{w}\right\}^T\ \underline{u} \qquad \text{(D-22)}$$

$$\frac{\partial}{\partial \lambda_0}\left(\frac{\partial \lambda_0}{\partial \underline{v}}\,\underline{u}\right) = \left\{-\frac{2}{ma^2}\left(\frac{\partial \underline{r}}{\partial \lambda_0} - \frac{3}{2}\,t\,\frac{\partial \underline{v}}{\partial \lambda_0}\right) + \frac{\sqrt{1-h^2-k^2}}{ma^2}\,\beta\left[\left(h\,\frac{\partial^2 X_1}{\partial \lambda_0 \partial h}\right.\right.\right.$$

$$\left.+ k\,\frac{\partial^2 X_1}{\partial \lambda_0 \partial k}\right)\underline{f} + \left(h\,\frac{\partial^2 Y_1}{\partial \lambda_0 \partial h} + k\,\frac{\partial^2 Y_1}{\partial \lambda_0 \partial k}\right)\underline{g}\Bigg]$$

$$\left. + \frac{\left(q\,\frac{\partial Y_1}{\partial \lambda_0} - p\,\frac{\partial X_1}{\partial \lambda_0}\right)}{ma^2\sqrt{1-h^2-k^2}}\,\underline{w}\right\}^T \underline{u} \tag{D-23}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial \lambda_0}{\partial \underline{v}}\,\underline{u}\right) = \left\{-\frac{2}{ma^2}\left(\frac{\partial \underline{r}}{\partial p} - \frac{3}{2}\,t\,\frac{\partial \underline{v}}{\partial p}\right) + \frac{\sqrt{1-h^2-k^2}}{ma^2}\,\beta\left[\left(h\,\frac{\partial X_1}{\partial h}\right.\right.\right.$$

$$\left.+ k\,\frac{\partial X_1}{\partial k}\right)\frac{\partial \underline{f}}{\partial p} + \left(h\,\frac{\partial Y_1}{\partial h} + k\,\frac{\partial Y_1}{\partial k}\right)\frac{\partial \underline{g}}{\partial p}\Bigg]$$

$$\left. + \frac{(qY_1 - pX_1)}{ma^2\sqrt{1-h^2-k^2}}\,\frac{\partial \underline{w}}{\partial p} - \frac{X_1}{ma^2\sqrt{1-h^2-k^2}}\,\underline{w}\right\}^T \underline{u} \tag{D-24}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial \lambda_0}{\partial \underline{v}}\,\underline{u}\right) = \left\{-\frac{2}{ma^2}\left(\frac{\partial \underline{r}}{\partial q} - \frac{3}{2}\,t\,\frac{\partial \underline{v}}{\partial q}\right) + \frac{\sqrt{1-h^2-k^2}}{ma^2}\,\beta\left[\left(h\,\frac{X_1}{h}\right.\right.\right.$$

$$\left.+ k\,\frac{\partial X_1}{\partial k}\right)\frac{\partial \underline{f}}{\partial q} + \left(h\,\frac{\partial Y_1}{\partial h} + k\,\frac{\partial Y_1}{\partial k}\right)\frac{\partial \underline{g}}{\partial q}\Bigg]$$

$$\left. + \frac{(qY_1 - pX_1)}{ma^2\sqrt{1-h^2-k^2}}\,\frac{\partial \underline{w}}{\partial q} + \frac{Y_1}{ma^2\sqrt{1-h^2-k^2}}\,\underline{w}\right\}^T \underline{u}$$

(D-25)

$$\frac{\partial}{\partial a}\left(\frac{\partial p}{\partial \underline{v}}\,\underline{u}\right) = \left(\frac{1}{Y_1}\,\frac{\partial Y_1}{\partial a} - \frac{1}{2a}\right)\frac{\partial p}{\partial \underline{v}}\,\underline{u} \tag{D-26}$$

$$\frac{\partial}{\partial h}\left(\frac{\partial p}{\partial \underline{v}}\,\underline{u}\right) = \left(\frac{1}{Y_1}\,\frac{\partial Y_1}{\partial h} + \frac{h}{1-h^2-k^2}\right)\frac{\partial p}{\partial \underline{v}}\,\underline{u} \tag{D-27}$$

$$\frac{\partial}{\partial k}\left(\frac{\partial \underline{p}}{\partial \underline{v}}\ \underline{u}\right) = \left(\frac{1}{Y_1}\ \frac{\partial Y_1}{k} + \frac{k}{1-h^2-k^2}\right) \frac{\partial \underline{p}}{\partial \underline{v}}\ \underline{u} \tag{D-28}$$

$$\frac{\partial}{\partial \lambda_0}\left(\frac{\partial \underline{p}}{\partial \underline{v}}\ \underline{u}\right) = \frac{1}{Y_1}\ \frac{\partial Y_1}{\partial \lambda_0}\ \frac{\partial \underline{p}}{\partial \underline{v}}\ \underline{u} \tag{D-29}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial \underline{p}}{\underline{v}}\ \underline{u}\right) = \frac{Y_1}{2ma^2\sqrt{1-h^2-k^2}} \left[2p\underline{w} + (1+p^2+q^2)\ \frac{\partial \underline{w}}{\partial p}\right]^T \underline{u} \tag{D-30}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial \underline{p}}{\partial \underline{v}}\ \underline{u}\right) = \frac{Y_1}{2ma^2\sqrt{1-h^2-k^2}} \left[2q\underline{w} + (1+p+q^2)\ \frac{\partial \underline{w}}{\partial q}\right]^T \underline{u} \tag{D-31}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u}\right) = \left(\frac{1}{X_1}\ \frac{\partial X_1}{\partial a} - \frac{1}{2a}\right) \frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u} \tag{D-32}$$

$$\frac{\partial}{\partial h}\left(\frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u}\right) = \left(\frac{1}{X_1}\ \frac{\partial X_1}{\partial h} + \frac{h}{1-h^2-k^2}\right) \frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u} \tag{D-33}$$

$$\frac{\partial}{\partial k}\left(\frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u}\right) = \left(\frac{1}{X_1}\ \frac{\partial X_1}{\partial k} + \frac{k}{1-h^2-k^2}\right) \frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u} \tag{D-34}$$

$$\frac{\partial}{\partial \lambda_0}\left(\frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u}\right) = \frac{1}{X_1}\ \frac{\partial X_1}{\partial \lambda_0} + \frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u} \tag{D-35}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u}\right) = \frac{X_1}{2ma^2\ \sqrt{1-h^2-k^2}} \left[2p\underline{w} + (1+p^2+q^2)\frac{\partial \underline{w}}{\partial p}\right]^T \underline{u} \tag{D-36}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial \underline{q}}{\partial \underline{v}}\ \underline{u}\right) = \frac{X_1}{2ma^2\sqrt{1-h^2-k^2}} \left[2q\underline{w} + (1+p^2+q^2)\frac{\partial \underline{w}}{\partial q}\right]^T \underline{u} \tag{D-37}$$

$$\frac{\partial^2 X_1}{\partial a \partial h} = \frac{1}{a}\frac{\partial X_1}{\partial h} - \frac{3}{2}\frac{mat}{r}\left[(1 - \frac{r}{a})(\beta + \frac{h^2\beta^3}{1-\beta}) + \frac{a}{r}\sin F(h\beta - \sin F)\right.$$

$$\left. + \frac{a}{r}\cos^2 F\right] \qquad \text{(D-38)}$$

$$\frac{\partial^2 X_1}{\partial h^2} = a\left[- \cos F(\beta + \frac{h^2\beta^3}{1-\beta}) + (k \sin F) - h \cos F)(\frac{2h\beta^3}{1-\beta})\right.$$

$$\left. - \frac{a}{r}\beta \cos F\right] - \frac{a}{r}\cos F \frac{\partial^2 X_1}{\partial F \partial h} + \frac{h\beta^3}{1-\beta}\frac{\partial^2 X_1}{\partial \beta \partial h}$$

$$- \left[\frac{a^2}{r}\cos F(1 + k \sin F - h \cos F) + a \sin F\right]\frac{\partial^2 X_1}{\partial r \partial h}$$

(D-39)

$$\frac{\partial^2 X_1}{\partial k \partial h} = a \sin F\ (\beta + \frac{h^2\beta^3}{1-\beta}) + \frac{a}{r}\sin F \frac{\partial^2 X_1}{\partial F \partial h} + \frac{k\beta^3}{1-\beta}\frac{\partial^2 X_1}{\partial \beta \partial h}$$

$$+ \left[\frac{a^2}{r}\sin F\ (1 + k \sin F - h \cos F - a \cos F\right]\frac{\partial^2 X_1}{\partial r \partial h}$$

(D-40)

$$\frac{\partial^2 X_1}{\partial \lambda_0 \partial h} = \frac{a}{r}\frac{\partial^2 X_1}{\partial F \partial h} + \frac{a^2}{r}(1+k \sin F - h \cos F)\frac{\partial^2 X_1}{\partial r \partial h} \qquad \text{(D-41)}$$

$$\frac{\partial^2 X_1}{\partial F \partial h} = a\left[(1 - \frac{r}{a})(\beta + \frac{h^2\beta^3}{1-\beta}) + \frac{a}{r}\sin F)(h\beta - \sin F)\right.$$

$$\left. + \frac{a}{r}\cos^2 F\right] \qquad \text{(D-42)}$$

$$\frac{\partial^2 X_1}{\partial \beta \partial h} = a\left[(k \sin F - h \cos F)(1 + \frac{h^2\beta^3\ (3-2\beta)}{(1-\beta)^2} - \frac{a}{r}h \cos F\right]$$

(D-43)

$$\frac{\partial^2 X_1}{\partial r \partial h} = \frac{a^2}{r^2}\cos F\ (h\beta - \sin F) \qquad \text{(D-44)}$$

$$\frac{\partial^2 X_1}{\partial a \partial k} = \frac{1}{a}\frac{\partial X_1}{\partial k} - \frac{3}{2}\frac{mat}{r}\left[(1-\frac{r}{a})\frac{hk\beta^3}{1-\beta} - \frac{a}{r}\cos F\ (2\sin F - h\beta)\right] \tag{D-45}$$

$$\frac{\partial^2 X_1}{\partial h \partial k} = \frac{\partial^2 X_1}{\partial k \partial h} \tag{D-46}$$

$$\frac{\partial^2 X_1}{\partial k^2} = \frac{ah\beta^3}{1-\beta}(2k\sin F - h\cos F) + \frac{a}{r}\sin F\frac{\partial^2 X_1}{\partial F \partial k} + \frac{k\beta^3}{1-\beta}\frac{\partial^2 X_1}{\partial \beta \partial k} + \left[\frac{a^2}{r}\sin F(1 + k\sin F - h\cos F) - a\cos F\right]\frac{\partial^2 X_1}{\partial r \partial k} \tag{D-47}$$

$$\frac{\partial^2 X_1}{\partial \lambda_0 \partial k} = \frac{a}{r}\frac{\partial^2 X_1}{\partial F \partial k} + \frac{a^2}{r}(1 + k\sin F - h\cos F)\frac{\partial^2 X_1}{\partial r \partial k} \tag{D-48}$$

$$\frac{\partial^2 X_1}{\partial F \partial k} = a\left[(1-\frac{r}{a})\frac{hk\beta^3}{1-\beta} - \frac{a}{r}\cos F(2\sin F - h\beta)\right] \tag{D-49}$$

$$\frac{\partial^2 X_1}{\partial \beta \partial k} = a\left[(k\sin F - h\cos F)\frac{hk\beta^2(3-2\beta)}{(1-\beta)^2} + \frac{a}{r}h\sin F\right] \tag{D-50}$$

$$\frac{\partial^2 X_1}{\partial r \partial k} = \frac{a^2}{r^2}\sin F\ (\sin F - h\beta) \tag{D-51}$$

$$\frac{\partial^2 Y_1}{\partial a \partial h} = \frac{1}{a}\frac{\partial Y_1}{\partial h} - \frac{3}{2}\frac{mat}{r}\left[(\frac{r}{a} - 1)\frac{kh\beta^3}{1-\beta} - \frac{a}{r}\sin F\ (k\beta - 2\cos F)\right] \tag{D-52}$$

$$\frac{\partial^2 Y_1}{\partial h^2} = -\frac{ak\beta^3}{1-\beta}(k \sin F - 2h \cos F) - \frac{a}{r} \cos F \frac{\partial^2 Y_1}{\partial F \partial h}$$

$$+ \frac{h\beta^3}{1-\beta} \frac{\partial^2 Y_1}{\partial \beta \partial h} - \left[\frac{a^2}{r} \cos F(1 + k \sin F - h \cos F)\right.$$

$$\left. + a \sin F\right] \frac{\partial^2 Y_1}{\partial r \partial h} \qquad \text{(D-53)}$$

$$\frac{\partial^2 Y_1}{\partial k \partial h} = a \left[-\frac{h\beta^3}{1-\beta} (2k \sin F - h \cos F) + \frac{a}{r} \beta \cos F\right]$$

$$+ \frac{a}{r} \sin F \frac{\partial^2 Y_1}{\partial F \partial h} + \frac{k\beta^3}{1-\beta} \frac{\partial^2 Y_1}{\partial \beta \partial h} + \left[\frac{a^2}{r} \sin F(1 + k \sin F\right.$$

$$\left. - h \cos F) - a \cos F\right] \frac{\partial^2 Y_1}{\partial r \partial h} \qquad \text{(D-54)}$$

$$\frac{\partial^2 Y_1}{\partial \lambda_0 \partial h} = \frac{a}{r} \frac{\partial^2 Y_1}{\partial F \partial h} + \frac{a^2}{r} (1 + k \sin F - h \cos F) \frac{\partial^2 Y_1}{\partial r \partial h} \qquad \text{(D-55)}$$

$$\frac{\partial^2 Y_1}{\partial F \partial h} = a\left[\left(\frac{r}{a} - 1\right) \frac{kh\beta^3}{1-\beta} - \frac{a}{r} \sin F \ (k\beta - 2\cos F)\right] \qquad \text{(D-56)}$$

$$\frac{\partial^2 Y_1}{\partial \beta \partial h} = a\left[- (k \sin F - h \cos F) \frac{kh\beta^2(3-2\beta)}{(1-\beta)^2} + \frac{a}{r} k \cos F\right] \qquad \text{(D-57)}$$

$$\frac{\partial^2 Y_1}{\partial r \partial h} = -\frac{a^2}{r^2} \cos F(k\beta - \cos F) \qquad \text{(D-58)}$$

$$\frac{\partial^2 Y_1}{\partial a \partial k} = \frac{1}{a}\frac{\partial Y_1}{\partial k} - \frac{3}{2}\frac{\text{mat}}{r}\left[(\frac{r}{a} - 1)(\beta + \frac{k^2\beta^3}{1-\beta}) + \right.$$

$$\left. + \frac{a}{r}\cos F\ (\cos F - k\beta) - \frac{a}{r}\sin^2 F\right] \qquad \text{(D-59)}$$

$$\frac{\partial^2 Y_1}{\partial h \partial k} = \frac{\partial^2 Y_1}{\partial k \partial h} \qquad \text{(D-60)}$$

$$\frac{\partial^2 Y_1}{\partial k^2} = a\left[- \sin F\ (\beta + \frac{k^2\beta^3}{1-\beta}) + (h\ \ \cos F - k \sin F)(\frac{2k\beta^3}{1-\beta})\right.$$

$$\left. - \frac{a}{r}\ \beta \sin F\right] + \frac{a}{r}\sin F\ \frac{\partial^2 Y_1}{\partial F \partial h} + \frac{k\beta^3}{1-\beta}\frac{\partial^2 Y_1}{\partial \beta \partial k}$$

$$+ \left[\frac{a^2}{r}\ \sin F(1 + k \sin F - h \cos F) - a \cos F\right]\frac{\partial^2 Y_1}{\partial r \partial k}$$

(D-61)

$$\frac{\partial^2 Y_1}{\partial \lambda_0 \partial k} = \frac{a}{r}\frac{\partial^2 Y_1}{\partial F \partial k} + \frac{a^2}{r}\ (1 + k \sin F - h \cos F)\frac{\partial^2 Y_1}{\partial r \partial k} \qquad \text{(D-62)}$$

$$\frac{\partial^2 Y_1}{\partial F \partial k} = a\left[(\frac{r}{a} - 1)(\beta + \frac{k^2\beta^3}{1-\beta}) + \frac{a}{r}\cos F\ (\cos F - k\beta)\right.$$

$$\left. - \frac{a}{r}\sin^2 F\right] \qquad \text{(D-63)}$$

$$\frac{\partial^2 Y_1}{\partial \beta \partial k} = a\left[(h \cos F - k \sin F)(1 + \frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}) - \frac{a}{r}\ k \sin F\right]$$

(D-64)

$$\frac{\partial^2 Y_1}{\partial r \partial k} = - \frac{a^2}{r^2} \sin F \; (\cos F - k\beta) \tag{D-65}$$

$$\frac{\partial \underline{f}}{\partial p} = - \frac{2}{1+p^2+q^2} \; [\, q \; \underline{g} + \underline{w}] \tag{D-66}$$

$$\frac{\partial \underline{f}}{\partial q} = \frac{2}{1+p^2+q^2} \; p \; \underline{g} \tag{D-67}$$

$$\frac{\partial \underline{g}}{\partial p} = \frac{2}{1+p^2+q^2} \; q \; \underline{f} \tag{D-68}$$

$$\frac{\partial \underline{g}}{\partial q} = - \frac{2}{1+p^2+q^2} \; [p \; \underline{f} - \underline{w}] \tag{D-69}$$

$$\frac{\partial \underline{w}}{\partial p} = \frac{2}{1+p^2+q^2} \; \underline{f} \tag{D-70}$$

$$\frac{\partial \underline{w}}{\partial q} = - \frac{2}{1+p^2+q^2} \; \underline{g} \tag{D-71}$$

$$\frac{\partial X_1}{\partial a} = \frac{\partial \underline{r}}{\partial a}^T \; \underline{f} \tag{D-72}$$

$$\frac{\partial X_1}{\partial \lambda_0} = \frac{\partial \underline{r}}{\partial \lambda_0}^T \; \underline{f} \tag{D-73}$$

$$\frac{\partial Y_1}{\partial a} = \frac{\partial \underline{r}}{\partial a}^T \underline{g} \tag{D-74}$$

$$\frac{\partial Y_1}{\partial \lambda_0} = \frac{\partial \underline{r}}{\partial \lambda_0}^T \underline{g} \tag{D-75}$$

$$\frac{\partial \dot{X}_1}{\partial a} = \frac{\partial \underline{v}}{\partial a}^T \underline{f} \tag{D-76}$$

$$\frac{\partial \dot{X}_1}{\partial \lambda_0} = \frac{\partial \underline{v}}{\partial \lambda_0}^T \underline{f} \tag{D-77}$$

$$\frac{\partial \dot{Y}_1}{\partial a} = \frac{\partial \underline{v}}{\partial a}^T \underline{g} \tag{D-78}$$

$$\frac{\partial \dot{Y}_1}{\partial \lambda_0} = \frac{\partial \underline{v}}{\partial \lambda_0}^T \underline{g} \tag{D-79}$$

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